1. The Richardson Number

1a. Flux Richardson Number

The ratio of the buoyant production term (Term III) and the mechanical production term (Term IV) is called the ** Flux Richardson Number (\( R_f \)). ** This number characterizes the thermal stability of the flow.
\( R_f = \frac{g}{\theta_v} \frac{(u' \theta'_v)}{(u'_i u'_j) \frac{\partial U}{\partial x_j}} \)  

(1)

The denominator consists of 9 terms. We assume horizontal homogeneity and neglect subsidence:

\( R_f = \frac{\frac{g}{\theta_v} (u' \theta'_v)}{(u'_i u'_j) \frac{\partial U}{\partial x_j} + (v' w') \frac{\partial V}{\partial z}} \)  

(2)

Remember, the denominator is usually negative.

- \( R_f > 0 \) for statically stable flows
- \( R_f < 0 \) for statically unstable flows
- \( R_f = 0 \) for statically neutral flows

At the critical value of \( R_f = +1 \), the mechanical production rate balances the buoyant consumption.

- \( R_f < +1 \) static stability is insufficient to prevent the mechanical generation of turbulence, flow is dynamically unstable. (Statically unstable flow is always dynamically unstable).
- \( R_f > +1 \) flow becomes laminar (dynamically stable)
- \( R_f = 0 \) for statically neutral flows

1b. Gradient Richardson Number

The value of the turbulent correlations could be expressed as being proportional to the lapse rate, and the turbulent momentum flux can be proportional to the wind gradient: \( w' \theta'_v \propto \frac{\partial \theta}{\partial z} \), \( w' u' \propto \frac{\partial U}{\partial z} \) and \( w' v' \propto \frac{\partial V}{\partial z} \). This is the basis of K-theory, that we will discuss later. When substituting into equation 4, we get the Gradient Richardson Number:

\[ R_i = \frac{\frac{g}{\theta_v} \frac{\partial \theta_v}{\partial z}}{\left( \frac{\partial U}{\partial z} \right)^2 + \left( \frac{\partial V}{\partial z} \right)^2} \]  

(3)

Laminar flow becomes turbulent when \( R_i < R_c \approx 0.21 \), and turbulent flow becomes laminar when \( R_i > R_T \approx 1 \). There is a hysteresis effect.
1c. Bulk Richardson Number

When measuring wind shear and temperature gradients, meteorologists approximate the gradients by measurements at discrete heights:

\[
R_B = \frac{g \frac{\Delta \theta}{\Delta z}}{\left( \frac{\Delta U}{\Delta z} \right)^2 + \left( \frac{\Delta V}{\Delta z} \right)^2} = \frac{g \Delta \theta_v \Delta z}{\bar{\theta}_v ((\Delta U)^2 + (\Delta V)^2)}
\]  

(4)

This is the form most frequently used. The values of the critical Richardson number don’t apply to these finite differences across thick layers. The thinner the layer, the closer the value to the theory.

2. The Obukhov Length

This is a very important parameter in the surface layer. Remember that in the surface layer we can assume constant flux with height (by definition). Let’s multiply the TKE equation by \((-kz/u^3_v)\) where \(k\) is the von Karman constant \(\approx 0.4\). We assume that all the turbulent fluxes are equal to their values at the surface, and focus only on the buoyant production and mechanical production terms.

\[
\ldots = -kzg \frac{w'\theta'_v}{\theta_v} \underbrace{u^3_s \frac{\bar{\theta}_v}{\theta_v}}_{III} + \frac{kz(u'_i u'_j)_{ss}}{u^3_s} \underbrace{\partial U_i}_{IV} + \ldots
\]  

(5)

Term III is assigned the symbol \(\zeta \equiv \frac{z}{L}\) where \(L\) is the Obukhov length.

\[
\zeta = \frac{z}{L} = \frac{-kzg w'\theta'_v}{u^3_s \bar{\theta}_v}
\]  

(6)

The Obukhov Length is given by

\[
L = \frac{-\bar{\theta}_v u^3_s}{k \theta'_v (w'_v)_{ss}}
\]  

(7)

Physical interpretation: proportional to the height above the surface at which buoyant factors first dominate over mechanical (shear) production of turbulence. Buoyant and shear production terms are approximately equal at \(z = -0.5L\). \(\zeta\) is a surface layer scaling parameter.

- \(\zeta < 0\) Unstable
• $\zeta > 0$ Stable

3. Dimensionless Gradients

Let’s look now at Term IV of the normalized TKE, with a system aligned with the mean wind, assuming horizontal homogeneity and neglecting subsidence. We use the definition of $u^*^2 = -(\bar{u}'w')_s$.

\[
\begin{align*}
... &= ... \frac{k_z}{u_s} \frac{\partial \bar{U}_i}{\partial z} + \ldots
\end{align*}
\]

(8)

Based on this term, we define a dimensionless wind shear:

\[
\phi_m = \frac{k_z}{u_s} \frac{\partial \bar{U}_i}{\partial z}
\]

(9)

This is used for surface-layer wind profiles and momentum fluxes. We use $\phi_m$ in similarity theory. By analogy, the dimensionless lapse rate $\phi_H$ and dimensionless humidity gradient $\phi_E$:

\[
\begin{align*}
\phi_H &= \frac{k_z}{q_s^L} \frac{\partial \bar{\theta}}{\partial z} \\
\phi_E &= \frac{k_z}{q_s^L} \frac{\partial \bar{q}}{\partial z}
\end{align*}
\]

(10)

(11)

All of these non-dimensional numbers are= 1 for Neutral Conditions.

4. Surface Layer Scaling

Recall when we spoke about K-Theory when discussing First Order Closure:

We can define a mixing length, $l$ by $l^2 = c\bar{z}$. In the surface layer eddies are limited by the earth’s surface. It is assumed that $l^2 = k^2 z^2$ where $k$ is the von Karman constant, so:

\[
K_E = K_H = K_m = l^2 \left| \frac{\partial \bar{U}}{\partial z} \right| = k^2 z^2 \left| \frac{\partial \bar{U}}{\partial z} \right|
\]

(12)
In reality, the origin of $z$ for a rough surface because the protrusion of roughness elements above the substrate surface displaces the entire flow upwards. We define the displaced height $z = Z - d$, where $d$ is the zero-displacement height and $Z$ is the height above the substrate surface (height above the actual ground surface).

Hence, for neutral conditions with no buoyancy, in the surface layer (assuming that the stress remains constant throughout the surface layer) we recall the friction velocity, choosing the x-axis appropriately, reduces to:

$$u^2_* = -\overline{uw'} = K_m \frac{\partial U}{\partial z} = k^2 z^2 \left( \frac{\partial U}{\partial z} \right)^2$$  \hspace{1cm} (13)

$$u_* = k z \left( \frac{\partial U}{\partial z} \right)$$  \hspace{1cm} (14)

Integration gives the famous log-wind profile for neutral conditions:

$$\frac{kU}{u_*} = \ln(z) + cnt$$  \hspace{1cm} (15)

5. Monin-Obukhov Similarity Theory

We can take into account the influence of buoyancy through the Richardson number $R_f$ or the Obukhov Length $L$. The way this is generally done is by taking the dimensionless gradients we had expressed before, which are equal to 1 in neutral conditions, and expressing them as functions of $\zeta$ for non-neutral conditions:

Based on this term, we define a dimensionless wind shear:

$$\phi_m(\zeta) = \frac{kz \partial \bar{U}_i}{u_* \partial z}$$  \hspace{1cm} (16)

$$\phi_H(\zeta) = \frac{kz \partial \bar{\theta}}{\theta_* \partial z}$$  \hspace{1cm} (17)

$$\phi_E(\zeta) = \frac{kz \partial \bar{q}}{q_* \partial z}$$  \hspace{1cm} (18)

(NOTE: substitute the variables for their values in the neutral BL and verify}
that these variables=1 in the neutral BL)

The forms of the $\phi$ functions have been extensively studied using observations from many experiments. Observations suggest that:

For $-5 < \zeta < 0$

$$\phi_m(\zeta) = (1 - 16\zeta)^{-1/4} \quad (19)$$
$$\phi_H(\zeta) = \phi_E(\zeta) = (1 - 16\zeta)^{-1/2} \quad (20)$$

For $0 < \zeta < 1$

$$\phi_m = \phi_H = \phi_E = 1 + 5\zeta \quad (21)$$

5a. **Integral forms of the flux-gradient relations**

i. **Wind** For the general, non-neutral case, the surface layer wind profile can be obtained by integrating equation 16:

$$\frac{\partial U}{\partial z} = \frac{u_* \phi_m}{k z} \quad (22)$$
$$U(z) = \frac{u_*}{k} \int_{z_0}^{z} \left( \frac{dz'}{z'} - \frac{dz'}{z'} + \phi_m \frac{dz'}{z'} \right)$$
$$= \frac{u_*}{k} \left[ ln\frac{z}{z_0} - \int_{z_0}^{z} (1 - \phi_m) \frac{dz'}{z'} \right]$$
$$= \frac{u_*}{k} \left[ ln\frac{z}{z_0} - \psi_m(\zeta) \right]$$

for $0 < \zeta <= 1$ Stable

$$\psi_m = -5\zeta \quad (23)$$

for $1 < \zeta$ Stable

$$\psi_m = -5ln(z/z_0) \quad (24)$$

for $\zeta < 0 \ (x = \phi_m^{-1} = (1 - 16\zeta)^{1/4})$ Unstable

$$\psi_m = 2ln\frac{1 + x}{2} + ln\frac{1 + x^2}{2} - 2tan^{-1}x + \frac{\pi}{2} \quad (25)$$

In this form, the effects of buoyancy can be interpreted as a deviation of the wind speed from the neutral value.
• In unstable conditions $0 < \phi < 1$ and $\psi > 0$

• In stable conditions $\phi > 0$ and $\psi < 0$

In the general case where we have two wind measurements at heights 1 and 2, we can extend the above expression to:

$$U_2 - U_1 = \frac{u_*}{k} \left[ \ln \frac{z_2}{z_1} - \psi_m(\zeta_2) + \psi_m(\zeta_1) \right]$$  \hspace{1cm} (26)

### ii. Temperature

In analogous form:

\[
\frac{k(\bar{\theta} - \theta_0)}{\theta_{SL}^*} = \ln \frac{z}{z_T} - \psi_H(\zeta) \hspace{1cm} (27)
\]

\[
\frac{k(\bar{\theta_v} - \theta_{v0})}{\theta_{vSL}^*} = \ln \frac{z}{z_T} - \psi_H(\zeta) \hspace{1cm} (28)
\]

Here $z_T$ is the surface scaling length for temperature. Formally $\theta = \theta_0$ at $z = z_T$, and $z_T$ is not necessarily equal to $z_0$. Notice that we are assuming the same nondimensional numbers apply to potential and virtual potential temperature.

- For $0 < \zeta <= 1$ Stable
  \[
  \psi_m = -5\zeta \hspace{1cm} (29)
  \]

- For $1 < \zeta$ Stable
  \[
  \psi_m = -5\ln(z/z_0) \hspace{1cm} (30)
  \]

- For $\zeta < 0$ ($y = \phi_1^{-1} = (1 - 16\zeta)^{1/2}$) Unstable
  \[
  \psi_H = 2\ln \frac{1 + y}{2} \hspace{1cm} (31)
  \]

In the general case where we have two temperature measurements at heights 1 and 2, we can extend the above expression to:

\[
\frac{k(\bar{\theta_2} - \bar{\theta_1})}{\theta_{SL}^*} = \ln \frac{z_2}{z_1} - \psi_H(\zeta_2) + \psi_H(\zeta_1) \hspace{1cm} (32)
\]

\[
\frac{k(\bar{\theta_{v2}} - \bar{\theta_{v1}})}{\theta_{vSL}^*} = \ln \frac{z_2}{z_1} - \psi_H(\zeta_2) + \psi_H(\zeta_1) \hspace{1cm} (33)
\]

### iii. Humidity

In analogous form:
\[
\frac{k(\overline{q} - q_0)}{q_s^{SL}} = \ln \frac{z}{z_q} - \psi_E(\zeta)
\]  
(34)

for \(0 < \zeta \leq 1\) Stable

\[\psi_m = -5\zeta\]  
(35)

for \(1 < \zeta\) Stable

\[\psi_m = -5 \ln(z/z_0)\]  
(36)

for \(\zeta < 0\) \((y = \phi_H^{-1} = (1 - 16\zeta)^{1/2})\) Unstable

\[\psi_E = 2 \ln \frac{1 + y}{2}\]  
(37)

In the general case where we have two humidity measurements at heights 1 and 2, we can extend the above expression to:

\[
\frac{k(\overline{q}_2 - \overline{q}_1)}{q_s^{SL}} = \ln \frac{z_2}{z_1} - \psi_E(\zeta_2) + \psi_E(\zeta_1)
\]  
(38)

Observations and theory suggest that \(\Phi_E = \Phi_H\) and \(\psi_E = \psi_H\) and \(z_q = z_T\)

5b. Calculating Fluxes using the Flux Profile Method

As we have shown before, if the stability and the flux or stress is known in advance, then the flux profile method can be used to solve directly for the wind speed or the potential temperature at any height. However, often these relationships are used in reverse, to estimate the flux knowing the mean wind or temperature profile. This is much more difficult. For example, \(u_*\) appears in a number of places, explicitly and hidden in \(L\), and \(L\) is a function of heat flux, which must be estimated from the temperature profile. Solving these equations involves an iterative approach.

Notice how we can use the above expressions to calculate the fluxes as:

\[u_* = \frac{k\overline{U_2} - \overline{U_1}}{\ln \frac{z_2}{z_1} - \psi_m(\zeta_2) + \psi_m(\zeta_1)}\]  
(39)

if \(z_1 = z_0\) then \(U_1 = 0\) and \(\psi_m(\zeta_1) = 0\)
\[ (w'\theta')_s = -u_* k (\bar{\theta}_2 - \bar{\theta}_1) \ln \frac{z_2}{z_1} - \psi_H(\zeta_2) + \psi_H(\zeta_1) \]  
(40)

\[ (w'\theta_v')_s = -u_* k (\bar{\theta}_v2 - \bar{\theta}_v1) \ln \frac{z_2}{z_1} - \psi_H(\zeta_2) + \psi_H(\zeta_1) \]  
(41)

\[ (w'q')_s = -u_* k (\bar{q}_2 - \bar{q}_1) \ln \frac{z_2}{z_1} - \psi_E(\zeta_2) + \psi_E(\zeta_1) \]  
(42)

Given the mean wind (U), pressure (P), humidity (q) and temperature (T) at a level z and at the "surface".

1. From the information given calculate the density of air \( \rho \) the latent heat \( L_v \) and \( \theta \) and \( \theta_v \).

2. Calculate \( u_* \), assuming neutral conditions.

3. Calculate \( \bar{w}'q' \)

4. Calculate \( \bar{w}'\theta', \bar{w}'\theta_v' \)

5. Calculate \( L \)

6. Begin iteration \( i \)

   (a) If \( L_i > 0 \) conditions are stable - calculate \( \psi_E = \psi_H \) and \( \psi_m \)

   (b) If \( L_i < 0 \) conditions are unstable - calculate \( \psi_E = \psi_H \) and \( \psi_m \)

   (c) If \( L_i = 0 \) conditions are neutral \( \psi_E = \psi_H = \psi_m = 0 \)

   (d) Re-calculate \( u_*, \bar{w}'q', \bar{w}'\theta' \) and \( L \) using the relationships that depend on stability \( \zeta \)

   (e) Calculate the difference in the fluxes \( \bar{w}'q', \bar{w}'\theta' \) between this iteration and the previous iteration. If the difference is large, continue to iterate until your answers converge.

6. Bulk Transfer Relations

For practical applications, we use drag and bulk transfer coefficients to relate fluxes to mean properties of the flow.
6a. Drag Coefficient

Using the relationship 23, and the definition of friction velocity \( u_\ast \), a drag coefficient \( C_D \) is defined as:

\[
C_D = \frac{(\overline{w'w''})^{1/2}}{U^2} = \frac{u_\ast^2}{U^2} = \frac{k^2}{\left[ \ln \frac{z}{z_0} - \psi_M(\zeta) \right]^2}
\]

(43)

\[
C_{DN} = \frac{k^2}{\left[ \ln \frac{z}{z_0} \right]^2}
\]

(44)

\[
C_D / C_{DN} = \left[ 1 - \frac{\psi_M(\zeta)}{\ln \frac{z}{z_0}} \right]^{-2}
\]

(45)

6b. Heat Transfer Coefficient

By analogy with the drag coefficient, a heat transfer coefficient \( C_H \) can be defined using the relationship 42 and the definition of \( \theta_s^{SL} = -\overline{w'\theta'_s}/u_\ast \)

\[
C_H = \frac{Q_H}{U(\theta_0 - \overline{\theta})} = \frac{(\overline{\theta'w'_s})}{U(\theta_0 - \overline{\theta})}
\]

(46)

\[
= \frac{(\overline{\theta'w'_s})}{-\frac{u_\ast}{k} \left[ \ln \frac{z}{z_0} - \psi_M(\zeta) \right] \frac{\theta_s^{SL}}{k} \left[ \ln \frac{z}{z_T} - \psi_H(\zeta) \right]}
\]

\[
= \frac{k^2}{\left[ \ln \frac{z}{z_0} - \psi_M(\zeta) \right] \left[ \ln \frac{z}{z_T} - \psi_H(\zeta) \right]}
\]

(47)

Where \( Q_H \) is the kinematic sensible heat which is the sensible heat divided by \( \rho C_p \)

\[
C_{HN} = \frac{k^2}{\left[ \ln \frac{z}{z_0} \right] \left[ \ln \frac{z}{z_T} \right]}
\]

(48)
\[
\frac{C_H}{C_{HN}} = \left[ \frac{1}{1 - \frac{\psi_M(\zeta)}{\ln \frac{z}{z_0}}} \right] \left[ \frac{1}{1 - \frac{\psi_H(\zeta)}{\ln \frac{z}{z_T}}} \right]
\]  \hspace{1cm} (49)

Values of \( C_{DN}/C_{HN} \) greater than one indicate the more efficient transfer of momentum than heat as the surface is rougher.

6c. Moisture Transfer Coefficient

By analogy with the heat transfer coefficient, a heat transfer coefficient \( C_E \) can be defined using the relationship 34 and the definition of \( q_s^{SL} = -\bar{w}' q_s' / u_s \)

\[
C_E = \frac{R}{U(q_0 - \bar{q})} = \frac{(q' w'_s)}{U(q_0 - \bar{q})}
\]  \hspace{1cm} (50)

\[
= \frac{(q' w'_s)}{-\frac{u_s}{k} \left[ \ln \frac{z}{z_0} - \psi_M(\zeta) \right] \frac{q_s^{SL}}{k} \left[ \ln \frac{z}{z_q} - \psi_E(\zeta) \right]}
\]  \hspace{1cm} (51)

\[
= \frac{k^2}{\left[ \ln \frac{z}{z_0} - \psi_M(\zeta) \right] \left[ \ln \frac{z}{z_q} - \psi_E(\zeta) \right]}
\]  \hspace{1cm} (51)

Where \( R \) is the kinematic vertical eddy moisture flux.

\[
C_{EN} = \frac{k^2}{\left[ \ln \frac{z}{z_0} \right] \left[ \ln \frac{z}{z_q} \right]}
\]  \hspace{1cm} (52)

\[
\frac{C_E}{C_{EN}} = \left[ \frac{1}{1 - \frac{\psi_M(\zeta)}{\ln \frac{z}{z_0}}} \right] \left[ \frac{1}{1 - \frac{\psi_E(\zeta)}{\ln \frac{z}{z_q}}} \right]
\]  \hspace{1cm} (53)

7. Aerodynamic Resistances

The drag, heat and mass transfer coefficients discussed above take into account both turbulent transfer and molecular transfer of a property between the surface and a reference height \( z \) in the surface layer. For some applications, it is more convenient to replace the transfer coefficients by "quasi-resistance" parameters.
In this approach, the linking of molecular transfer in the interfacial layer and turbulent transfer in the surface layer is simplified. This relates to the additive property of resistances in series. By analogy to Ohm’s law (resistance = potential difference / current). For any concentration difference \((\gamma_0 - \gamma)\) and flux \(F_s\)

\[
  r_a = (\gamma_0 - \gamma) / F_s \tag{54}
\]

\(r_a\) has dimensions of \(\text{sm}^{-1}\). The reciprocal \(r_a^{-1}\) is the conductance.

7a. Momentum

From the definition of \(C_D\) (equation 43, we define the bulk aerodynamic resistance to the transfer of momentum from a level \(z\) to the surface \(z = z_0\) as:

\[
  r_{aM} = \frac{\rho (u(z) - u(z_0))}{\tau_s} = \frac{u(z)}{u_2^*} = \left( C_D u(z) \right)^{-1} \tag{55}
\]

As \(C_D\) increases or \(u(z)\) increases, the resistance increases.

7b. Heat

From the definition of \(C_H\), we define the bulk aerodynamic resistance to the transfer of heat from the surface \(z = z_0\) to a level \(z\) as:

\[
  r_{aH} = \frac{\theta_0 - \theta}{H_0} = \left( C_H u(z) \right)^{-1} \tag{56}
\]

7c. Moisture

From the definition of \(C_E\), we define the bulk aerodynamic resistance to the transfer of moisture from the surface \(z = z_0\) to a level \(z\) as:

\[
  r_{aE} = \frac{(q_0 - \theta)}{E_0} = \left( C_E u(z) \right)^{-1} \tag{57}
\]

It is important to note that under near-neutral conditions, the resistance for moisture and heat is higher than for momentum.

Figure 1: Figure 3.8 Garrat
The surface values $\theta_0$ and $q_0$ must be estimated to use the expressions for the bulk aerodynamic resistance to sensible and latent heat exchange:

\[
  r_{aH} = \frac{\rho c_p (\theta_0 - \theta)}{H_0} \tag{58}
\]
\[
  r_{aV} = \frac{\rho (q_0 - q)}{E_0} \tag{59}
\]