Averaging the Conservation Relations (Chapter 4)

Basic problem need to address before discretization is how to represent processes on sub-grid scale.

Here the goal is to develop the mathematical formalism to describe this process

Use Reynold's averaging procedure

$$\phi = \bar{\phi} + \phi''$$

grid volume average

Sub-grid scale processes.

What we're interested in are modeled fluxes. So, for example vertical heat flux $(w\theta)$

$$w = \bar{w} + w'' \qquad \theta = \bar{\theta} + \theta''$$

$$\overline{w\theta} = \overline{(\bar{w} + w'')(\bar{\theta} + \theta'')}$$

$$= \overline{\bar{w}\bar{\theta}} + \overline{w''\bar{\theta}} + \overline{\theta''\bar{w}} + \overline{w''\theta''}$$

$$= \bar{w}\bar{\theta} + \overline{w''}\bar{\theta} + \overline{\theta''}\bar{w} + \overline{w''\theta''}$$

Assumptions:

- Terms which average subgrid scale processes $= 0$ (e.g. $\overline{w''} = 0$)
- The average of grid-vol average terms is $=$ grid vol. average. (e.g. $\overline{\overline{w}} = \overline{w}$)
- The avg. product of subgrid-scale fluxes is <u>not</u> zero ($\overline{w''\theta''} \neq 0$)

$$\text{So.. } \overline{w\theta} = \overline{w}\,\overline{\theta} + \overline{w''\theta''}$$

Grid-scale heat flux (resolvable flux)

Sub-grid scale heat flux (unresolvable flux)

$\Rightarrow$ Must parameterize!

For example, for equation of motion, just considering advective term:

$$\overline{\frac{\partial}{\partial t}(\overline{u_i} + u_i'') = -(u_j + u_j'')\frac{\partial}{\partial x_j}(\overline{u_i} + u_i'')}$$

$$\frac{\partial \overline{u_i}}{\partial t} = -\overline{u_j}\frac{\partial \overline{u_i}}{\partial x_j} - \overline{u_j''\frac{\partial u_i''}{\partial x_j}}$$

Another wrinkle we have to consider – and add even more notation to keep track of!

Mesoscale perturbation

$\Rightarrow$ NOT the same as sub-grid scale flux (with " notation)

Modification to the base synoptic state on the mesoscale, so modifies the grid volume avg.

$$\underbrace{\bar{\phi}}_{\text{Grid volume avg.}} = \underbrace{\phi_0}_{\text{Base synoptic state}} + \underbrace{\phi'}_{\text{Mesoscale perturbations}}$$

~~*~~ Boussinesq approx $\rightarrow$ Assume that in eq. of motion density can be replaced by it's mean value (ie. base synoptic state) everywhere EXCEPT in the vertical.

$\Rightarrow$ A very common assumption in numerical and linear models (e.g. derivation of atmospheric waves)

Boussinesq approx.

Equation of motion (neglecting sub-grid scale flux terms)

$$\frac{\partial \bar{u}_i}{\partial t} = -\bar{u}_j \frac{\partial \bar{u}_i}{\partial x_j} - (\alpha_0 + \alpha') \frac{\partial p_0}{\partial x_i} - (\alpha_0 + \alpha') \frac{\partial p'}{\partial x_i}$$

$$- g\delta_{i3} - 2\varepsilon_{ijk}\Omega_j \bar{u}_k$$

where $\bar{p} = p_0 + p'$ $\quad \alpha = \alpha_0 + \alpha'$

Vertical momentum equation (from book):

- No averaging or perturbation applied
- Lagrangian derivative for w
- Neglect vertical Coriolis force

$$\rho \frac{dw}{dt} = -\frac{\partial p}{\partial z} - \rho g. \tag{2.72}$$

For the horizontal components of the eq. of motion (u & v), α' is neglected, so PGF term is:

$$-\alpha_0 \frac{\partial (p_0 + p')}{\partial x_i} = -\alpha_0 \frac{\partial \bar{p}}{\partial x_i} \quad \text{for } i = 1, 2$$

But for the vertical eq. → PGF term.

$$-\left(\alpha_0 \frac{\partial p_0}{\partial z} + \alpha' \frac{\partial p_0}{\partial z}\right) - \left(\alpha_0 \frac{\partial p'}{\partial z} + \alpha' \frac{\partial p'}{\partial z}\right)$$

$$\mathbf{PGF}_z = -\alpha_0\left(1 + \frac{\alpha'}{\alpha_0}\right) \frac{\partial p_0}{\partial z} - \alpha_0\left(1 + \frac{\alpha'}{\alpha_0}\right) \frac{\partial p'}{\partial z}$$

Using fact that

$$\frac{\partial p_0}{\partial z} = -\frac{g}{\alpha_0} \quad \text{and} \quad \frac{\alpha'}{\alpha_0} \text{ \underline{is} relevant for vertical}$$

(Hydrostatic)

$$\textbf{PGF}_z = -\alpha_0\left(1+\frac{\alpha'}{\alpha_0}\right)\left(\frac{-g}{\alpha_0}\right) - \alpha_0\left(1+\frac{\alpha'}{\alpha_0}\right)\frac{\partial p'}{\partial z}$$

$$\textbf{PGF}_z = +g + \frac{\alpha'}{\alpha_0}g - \alpha_0\frac{\partial p'}{\partial z} - \cancel{\frac{\alpha'}{\alpha_0}\frac{\partial p'}{\partial z}} \text{ to zero}$$

Substituting back into vertical eq. of motion:

$$\frac{\partial \bar{w}}{\partial t} = -\bar{u}_j\frac{\partial \bar{w}}{\partial x_j} - \alpha_0\frac{\partial p'}{\partial z} + \frac{\alpha'}{\alpha_0}g + 2\bar{u}\Omega\cos\phi$$

→ Vert. eq. of motion with Boussinesq approx.

→ Equivalent to (4-15) in book neglecting subgrid scale flux terms.

How to equate $\theta'/\theta_0 \approx \frac{\alpha'}{\alpha_0}$

$$\theta = \frac{p}{\rho R}\left(\frac{p_s}{p}\right)^{R/c_p} = \frac{p\alpha}{R}\left(\frac{p_s}{p}\right)^{R/c_p}$$

$$\ln\theta = \ln\left(\frac{p\alpha}{R}\right) - \frac{R}{c_p}\ln p + C$$

Using fact $c_p = c_v + R$

$$\ln\theta = \frac{c_v}{c_p}\ln p + \ln\alpha + C$$

Define $\gamma = c_p/c_v$

Linearizing about a synoptic state

$$\ln\left(\theta_0\left(1+\frac{\theta'}{\theta_0}\right)\right) = \frac{1}{\gamma}\ln\left(p_0\left(1+\frac{p'}{p_0}\right)\right) + \ln\left(\alpha_0\left(1+\frac{\alpha'}{\alpha_0}\right)\right)$$

Since $\ln(1+\varepsilon) \approx \varepsilon$

$$\frac{\theta'}{\theta_0} \approx \frac{1}{\gamma}\frac{p'}{p_0} + \frac{\alpha'}{\alpha_0}$$

Assuming density changes due to pressure changes much smaller due to T changes

$$\theta'/\theta_0 \approx \alpha'/\alpha_0$$

The Boussinesq approximation is typically expressed in terms of potential temperature

$$\frac{\alpha'}{\alpha_0} \simeq \frac{T'}{T_0} - \frac{p'}{p_0}$$

$$\frac{\alpha'}{\alpha_0} \simeq \frac{\theta'}{\theta_0} - \cancel{\frac{c_v p'}{c_p p_0}}$$

Neglected.

For buoyancy wave motions, density fluctuations due to pressure are small compared to temperature changes.
Fluctuations in pressure cause sound waves, which are pretty unimportant for the mesoscale ...,

So then vertical eq. of motion is:

$$\frac{\partial \bar{w}}{\partial t} = \bar{u}_j \frac{\partial \bar{w}}{\partial x_j} - \alpha_0 \frac{\partial p'}{\partial z} + \frac{\theta'}{\Theta_0} g + 2u\Omega\cos\phi$$

Equivalent to 4-20 (without subgrid scale perturbations.)

$$\frac{\theta'}{\Theta_0} g$$

Buoyancy term (B)

The same term as in equation for convective available potential energy!

$$B = -\frac{\rho'}{\overline{\rho}} g \approx \left(\frac{T_v'}{\overline{T}_v} - \frac{p'}{\overline{p}} \right) g, \tag{2.77}$$

$$B \approx \frac{T_{v_p} - T_{v_{env}}}{T_{v_{env}}} g, \tag{2.78}$$

When hydrometeors are present and assumed to be falling at their terminal velocity, the downward acceleration due to drag from the hydrometeors is equal to gr_h, where r_h is the mass of hydrometeors per kg of air (maximum values of r_h within a strong thunderstorm updraft typically are 8–18 g kg^{-1}). The effect of this *hydrometeor loading* on an air parcel can be incorporated into the buoyancy; for example, we can rewrite (2.77) as

$$\begin{aligned} B &\approx \left(\frac{T_v'}{\overline{T}_v} - \frac{p'}{\overline{p}} - r_h \right) g = \left[\frac{\theta_v'}{\overline{\theta}_v} + \left(\frac{R}{c_p} - 1 \right) \frac{p'}{\overline{p}} - r_h \right] g \\ &= \left[\frac{\theta_\rho'}{\overline{\theta}_\rho} + \left(\frac{R_d}{c_p} - 1 \right) \frac{p'}{\overline{p}} \right] g, \end{aligned} \tag{2.79}$$