Idealized dynamical perspective of baroclinic instability: Eady Problem

Conceptual setup

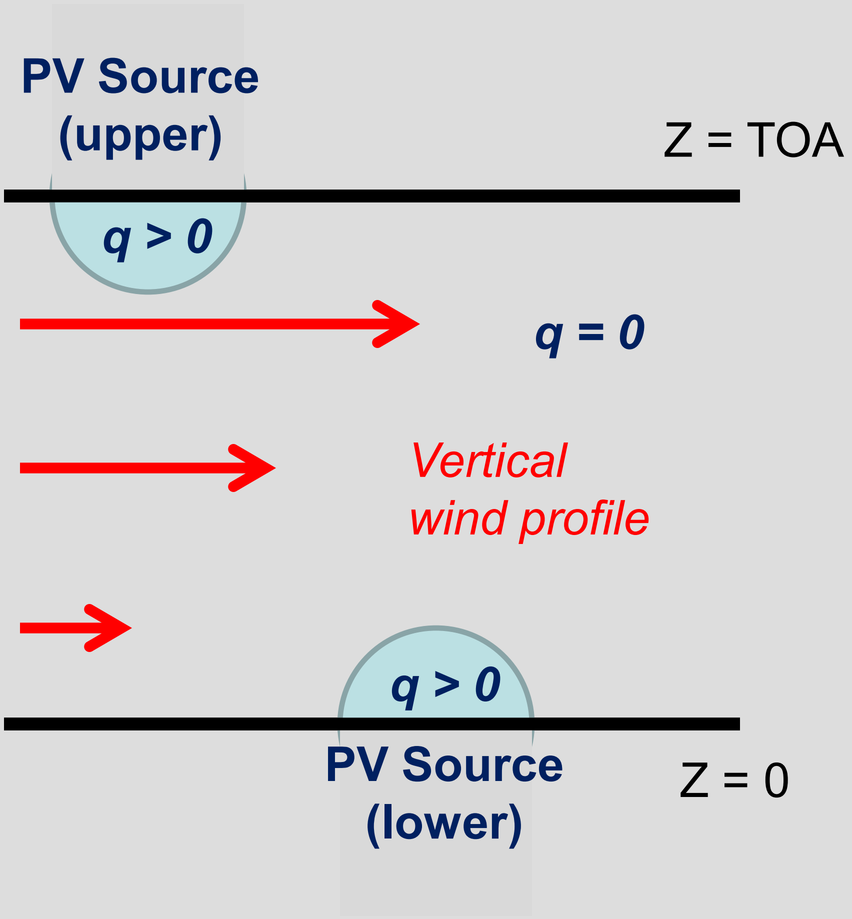
Frictionless, hydrostatic, f-plane in vertical

Consider 2-D plane in x-z with rigid lid on top and bottom

Linear vertical wind shear profile (implies baroclinicity, why???)

Basic state potential vorticity (q) of zero in the interior of domain

Initial sources of PV on upper and lower boundaries.



Physical implication of solution

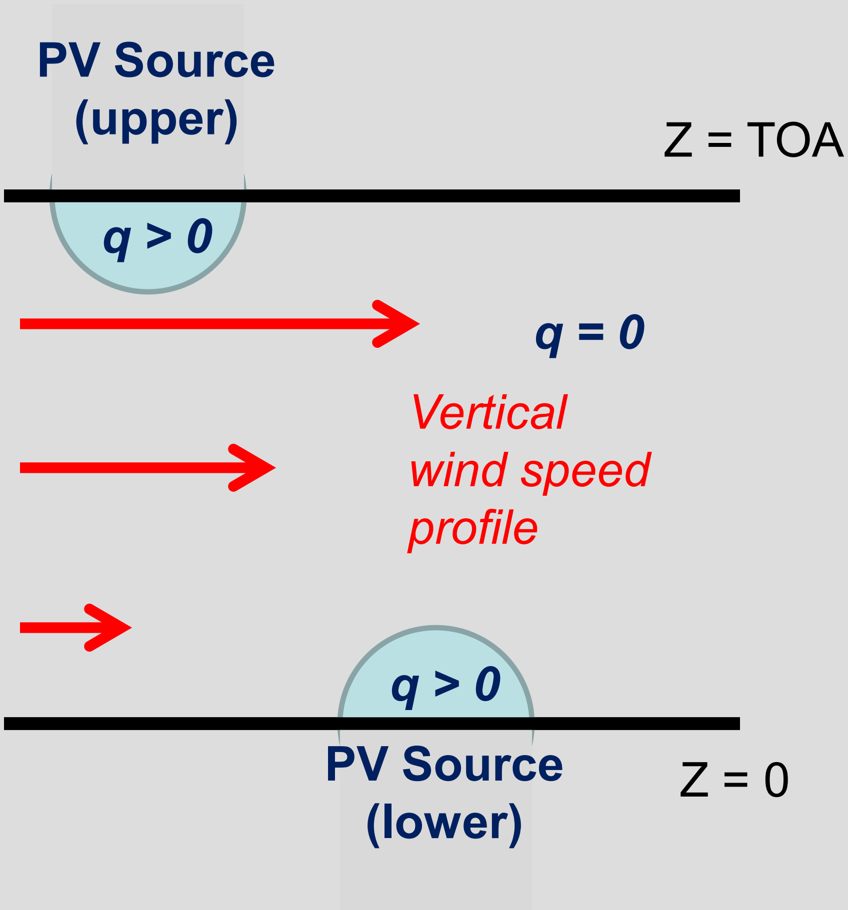
Assuming a wave solution, a dispersion relation can be constructed for phase speed

Because there are real and imaginary parts to the phase speed solution, possibility for exponential growth

Amplification = PV sources are within a preferred range of distance that they begin to interact with each other.

Growing solutions only occur for wavelengths greater than ~2500 km
Short-wave cutoff

Fastest growing waves about ~4000 km
Most unstable Eady mode.



Characteristics of most unstable Eady mode

Tilted structure to highs and lows
(as seen by streamfunction solution)

Isobars cross isotherms

Maximum southerly winds and rising motion ahead of upper-level low

Maximum northerly winds and sinking motion ahead of upper-level high

Just about what you get for a real mid-latitude cyclone in the mature to occluded stage!

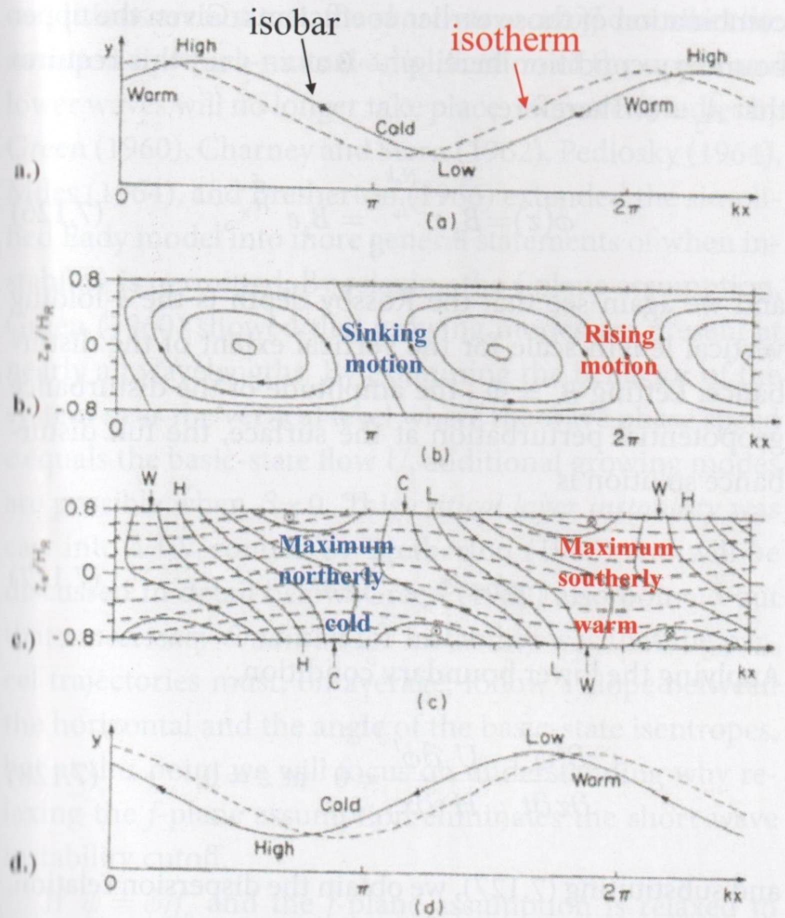


Figure 7.9. Structure of most unstable Eady wave: plan view of isobar and stream line at (a) the top of the domain ($z = H$) and (d) the lower boundary ($z = 0$), and cross section in x and z of (b) ageostrophic streamfunction, and (c) meridional wind (solid contours) and potential temperature (dashed contours; from Gill 1982).

Now…we want the more quantitative, thorough consideration of the analytic model just described…

Hang on, folks, here comes the heavy math):0

Defining a basic, mean state of the atmosphere in geostrophic and hydrostatic balance

$$\overline{u}(y,z) = -\frac{1}{f_o}\frac{\partial \Phi}{\partial y}$$

Zonal mean geostrophic wind: just depends on equator to pole temperature gradient

$$0 = -\frac{\partial \Phi}{\partial z} + \frac{R\overline{T}}{H}$$

Hydrostatic in vertical

$$\Phi = \Phi(y,z)$$

Geopotential varies only in vertical and meridional directions

$$\overline{v} = 0$$

Zero mean meridional wind

$$\overline{w} = 0$$

Zero vertical velocity

Geostrophic potential vorticity (Q) equation

$$Q = \frac{1}{f_o}\nabla^2\Phi + f + \frac{1}{\rho}\frac{\partial}{\partial z}\left(\frac{f_o}{N^2}\rho\frac{\partial\Phi}{\partial z}\right)$$

$$\nabla^2\Phi = \frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y}$$

Linearize PV equation using Reynold's averaging, considering:

$$\Phi = \overline{\Phi} + \Phi' \qquad u_g = \overline{u_g} + u_g{}'$$

$$v_g = v_g{}' \qquad w = w'$$

Linearize the PV equation

Linearized PV: Local time tendency + Advection + eddy conversion from the large-scale meridional PV gradient

$$\left(\frac{\partial}{\partial t}+\overline{u_g}\frac{\partial}{\partial x}\right)q' + \frac{1}{f_o}\frac{\partial \Phi'}{\partial x}\frac{\partial \overline{Q}}{\partial y} = 0$$

Change in large-scale meridional PV gradient (will be assumed zero in Eady problem)

$$\frac{\partial \overline{Q}}{\partial y} = \frac{\partial^2 \overline{u}}{\partial y^2} + \beta - \frac{1}{\rho}\frac{\partial}{\partial z}\left(\frac{f_o}{N^2}\rho\frac{\partial \overline{u}}{\partial z}\right)$$

Perturbation PV:

$$q' = \frac{1}{f_o}\nabla^2\Phi' + \frac{1}{\rho}\frac{\partial}{\partial z}\left(\frac{f_o}{N^2}\rho\frac{\partial \Phi'}{\partial z}\right)$$

Linearize the thermodynamic energy equation

Thermodynamic energy equation (adiabatic)

$$\frac{d}{dt}\left(\frac{\partial \Phi}{\partial z}\right) + wN^2 = 0$$

Linearized thermodynamic

$$\left(\frac{\partial}{\partial t} + \overline{u_g}\frac{\partial}{\partial x}\right)\frac{\partial \Phi'}{\partial z} + v_g{}'\frac{\partial}{\partial y}\left(\frac{\partial \overline{\Phi}}{\partial z}\right) + w'N^2 = 0$$

Applying geostrophic wind relation to second term on RHS:

$$\left(\frac{\partial}{\partial t} + \overline{u_g}\frac{\partial}{\partial x}\right)\frac{\partial \Phi'}{\partial z} - \frac{\partial \Phi'}{\partial x}\frac{\partial \overline{u_g}}{\partial z} + w'N^2 = 0$$

Rigid lid boundary conditions at z=0 and z=z_{top} (no vertical motion)

$$\left(\frac{\partial}{\partial t} + \overline{u_g}\frac{\partial}{\partial x}\right)\frac{\partial \Phi'}{\partial z} - \frac{\partial \Phi'}{\partial x}\frac{\partial \overline{u_g}}{\partial z} = 0$$

Assume a wave solution to linearized PV, thermodynamic equations

Express the geopotential anomaly as the product of a structure function and waveform with wavenumber k (in zonal direction):

$$\Phi' = \hat{\Phi}(y,z)\exp\left[ik(x-ct)\right]$$

Result after substitution into the equation for the linearized PV equation:

$$\left(\overline{u}-c\right)\left[\frac{\partial^2\hat{\Phi}}{\partial y^2}-k^2\hat{\Phi}+\frac{1}{\rho}\frac{\partial}{\partial z}\left(\frac{f_o^2}{N^2}\rho\frac{\partial\hat{\Phi}}{\partial z}\right)\right]+\hat{\Phi}\frac{\partial\overline{Q}}{\partial y}=0$$

Result after substitution into the equation for the linearized thermodynamic equation:

$$\left(\overline{u}-c\right)\frac{\partial\hat{\Phi}}{\partial z}-\hat{\Phi}\frac{\partial\overline{u}}{\partial z}=0$$

Need an analytic solution to the geopotential height structure function

Structure function for linearized PV equation: ***will apply in the interior of the domain of the Eady model***

$$\left(\overline{u}-c\right)\left[\frac{\partial^2\hat{\Phi}}{\partial y^2}-k^2\hat{\Phi}+\frac{1}{\rho}\frac{\partial}{\partial z}\left(\frac{f_o^2}{N^2}\rho\frac{\partial\hat{\Phi}}{\partial z}\right)\right]+\hat{\Phi}\frac{\partial\overline{Q}}{\partial y}=0$$

Structure function for linearized thermodynamic equation: ***will apply in the bottom and top boundary conditions of the Eady model***

$$\left(\overline{u}-c\right)\frac{\partial\hat{\Phi}}{\partial z}-\hat{\Phi}\frac{\partial\overline{u}}{\partial z}=0$$

The eigenvalue solutions of these geopotential height structure function equations can be used to:

1. **Determine if flow is (exponentially unstable)**
2. **Find most unstable wavenumber solution and e-folding time**

Eady problem creates one possible <u>simplified version</u> of these equations based on a set of predetermined assumptions and then arrives at a dispersion relation (equation for c) to assess stability.

Rayleigh and Fjortoft's theorems: Define requisite conditions for existence of unstable modes with (meridional) shear ($c_i > 0$)

	Barotropic instability	Baroclinic instability
Rayleigh's condition	The meridional PV gradient must change sign in the domain	The absolute meridional vorticity gradient must change sign in the domain
Fjortoft's condition	Meridional PV gradient must be positively correlated with zonal wind.	Meridional vorticity gradient must be positively correlated with zonal wind.

We're skipping over the formalized mathematical proofs of these theorems (see Gill and Pedlosky texbooks)

Idealized dynamical perspective of baroclinic instability: Eady Problem

<u>Conceptual setup</u>

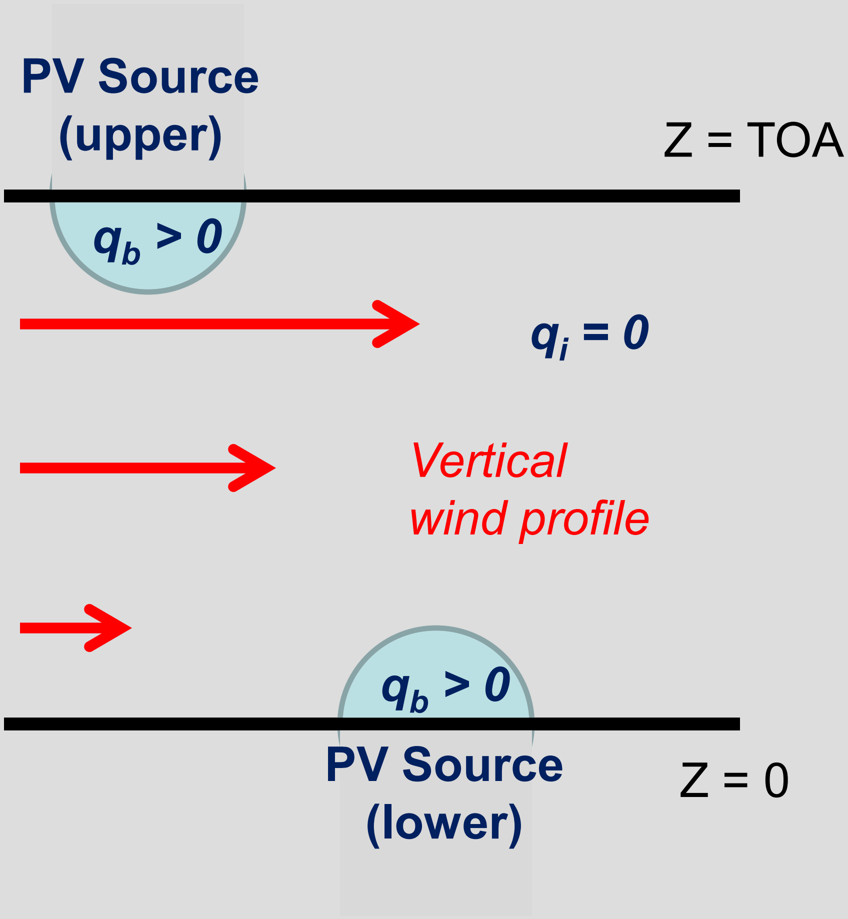
Frictionless, hydrostatic, f-plane in vertical

Consider 2-D plane in x-z with rigid lid on top and bottom

Linear vertical wind shear profile that implies meridional temperature gradient by thermal wind

Basic state potential vorticity (q) of zero in the interior of domain

Initial sources of PV on upper and lower boundaries.



Are Rayleigh and Fjortoft's conditions satisfied for Eady model?

Total PV (q) = Interior PV(q_i) + Boundary PV (q_b)

$$Q = \frac{1}{f_o}\nabla^2\Phi + f + \frac{1}{\rho}\frac{\partial}{\partial z}\left(\frac{f_o}{N^2}\rho\frac{\partial\Phi}{\partial z}\right)$$

The boundary PV anomalies are considered mathematically as delta functions on the top and bottom boundaries. The difference in sign arises because of the vertical integration of the last term on the RHS of previous equation.

$$q = q_i + \frac{f_o}{N^2}\frac{\partial\overline{\Phi}}{\partial z}\delta(z=0) - \frac{f_o}{N^2}\frac{\partial\overline{\Phi}}{\partial z}\delta(z=z_{top})$$

Recalling that there's no PV in model interior, meridional PV gradient is:

$$\frac{\partial q}{\partial y} = \frac{f_o^2}{N^2}\frac{\partial}{\partial z}\left(\frac{1}{f_o}\frac{\partial\overline{\Phi}}{\partial y}\delta(z=0)\right) - \frac{f_o^2}{N^2}\frac{\partial}{\partial z}\left(\frac{1}{f_o}\frac{\partial\overline{\Phi}}{\partial y}\delta(z=z_{top})\right)$$

Utilizing geostrophic wind relationships, meridional PV gradient can be expressed as function of vertical wind shear of zonal wind:

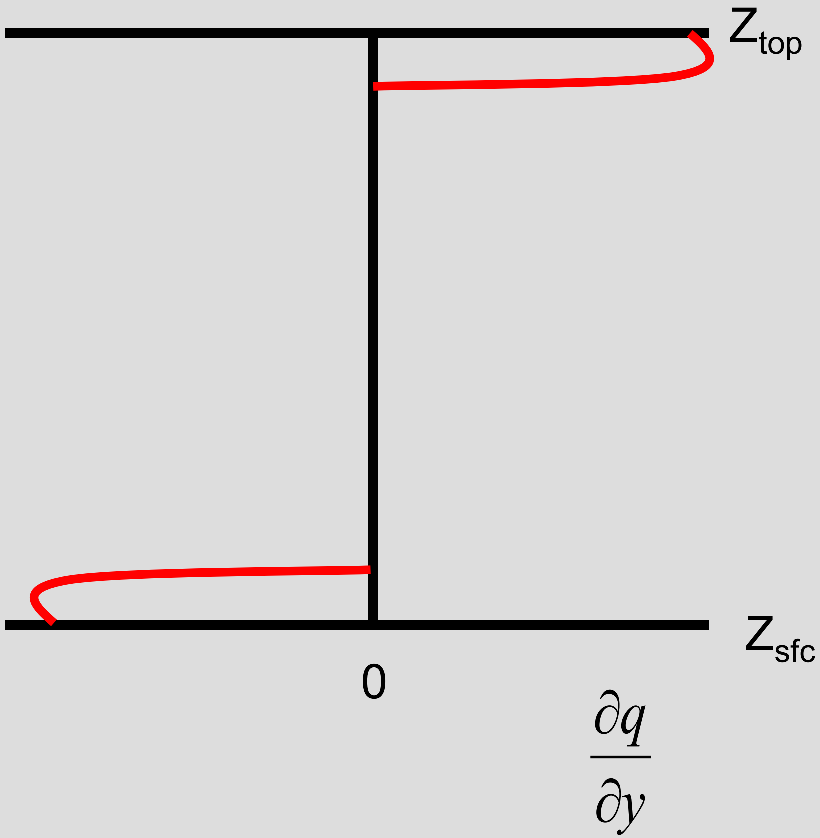
$$\frac{\partial q}{\partial y} = \frac{f_o^2}{N^2}\frac{\partial\overline{u}}{\partial z}\delta(z=z_{top}) - \frac{f_o^2}{N^2}\frac{\partial\overline{u}}{\partial z}\delta(z=0)$$

Meridional PV gradient equation is what is needed to assess whether Rayleigh and Fjortoft's conditions for baroclinic instability are satisfied for Eady problem:

$$\frac{\partial q}{\partial y} = -\frac{f_o^2}{N^2}\frac{\partial u}{\partial z}\delta(z=0) + \frac{f_o^2}{N^2}\frac{\partial u}{\partial z}\delta(z=z_{top})$$

Rayleigh condition satisfied because meridional vorticity gradient changes sign in the domain

Fjortoft's condition satisfied because the meridional vorticity gradient is positively correlated with zonal wind profile (since u is linearly increasing from a value of zero at the surface)



Simplified PV and thermodynamic structure equations for Eady problem

Potential vorticity: interior

$$\left(\bar{u}-c\right)\left[\frac{\partial^2\hat{\Phi}}{\partial y^2}-k^2\hat{\Phi}+\frac{1}{\rho}\frac{\partial}{\partial z}\left(\frac{f_o^2}{N^2}\rho\frac{\partial\hat{\Phi}}{\partial z}\right)\right]+\hat{\Phi}\frac{\partial\bar{Q}}{\partial y}=0$$

Thermodynamic: boundaries

$$\left(\bar{u}-c\right)\frac{\partial\hat{\Phi}}{\partial z}-\hat{\Phi}\frac{\partial\bar{u}}{\partial z}=0$$

Now, both with Eady problem assumptions applied…

$$\left(\Lambda z-c\right)\left(\frac{\partial^2\hat{\Phi}}{\partial y^2}-k^2\hat{\Phi}+\varepsilon\frac{\partial^2\hat{\Phi}}{\partial z^2}\right)=0$$

$$\left(\Lambda z-c\right)\frac{\partial\hat{\Phi}}{\partial z}-\hat{\Phi}\Lambda=0$$

$$\Lambda=\frac{\partial\bar{u}}{\partial z}\qquad\varepsilon=\frac{f_o^2}{N^2}$$

Since the structure equations are homogeneous in y and z directions, we can assume (where l is a wavenumber in y):

$$\hat{\Phi}(y,z) = \sin ly \cdot \hat{\Psi}(z)$$

Rewriting the structure equations invoking this additional assumption, we can further simplify them to an analytic form we can solve to get the dispersion relation (by solving an second order differential equation):

<u>Potential vorticity: interior</u>

$$\left(\Lambda z - c\right)\frac{\partial^2 \hat{\Psi}}{\partial z^2} - \alpha^2 \hat{\Psi} = 0$$

<u>Thermodynamic: boundaries</u>

$$\left(\Lambda z - c\right)\frac{\partial \hat{\Psi}}{\partial z} - \hat{\Psi}\Lambda = 0$$

$$\Lambda = \frac{\partial \overline{u}}{\partial z} \qquad \alpha^2 = \frac{k^2 + l^2}{f_o^2 / N^2}$$

Deriving a dispersion relation for Eady system

Neglect trivial solutions associated with

$$\Lambda z - c = 0$$

Then the Eady PV structure equation simplifies to:

$$\frac{\partial^2 \hat{\Psi}}{\partial z^2} - \alpha^2 \hat{\Psi} = 0$$

The solution to this differential equation is:

$$\hat{\Psi}(z) = A \sinh(\alpha z) + B \cosh(\alpha z)$$

Subject to the boundary conditions (by thermodynamic equation) at z=0 and $z=z_{top}=H$:

$$\left(\Lambda z - c\right)\frac{\partial \hat{\Psi}}{\partial z} - \hat{\Psi}\Lambda = 0$$

$$-c\alpha A - B\Lambda = 0$$ Bottom boundary

$$\alpha(\Lambda H - c)(A\cosh\alpha H + B\sinh\alpha H) - \Lambda(A\sin\alpha H + B\cosh\alpha H) = 0$$ Top boundary

This equation set defines a homogeneous system with two unknowns that can be solved to get a solution for phase speed (c), or the dispersion relation:

$$c = \frac{\Lambda H}{2} \pm \frac{\Lambda H}{2}\left[1 - \frac{4\cos\alpha H}{\alpha H\sin\alpha H} + \frac{4}{\alpha^2 H^2}\right]^{1/2}$$

If the bracketed square root term is less than zero, the solution is exponentially unstable because $c_i > 0$. The unstable solutions will define the range of conditions where we have spontaneously amplifying baroclinic waves.

These conditions will be dependent on all the environmental parameters assumed as "constants" in the Eady problem (vertical wind shear, stability, Coriolis parameter).

Defining the range of unstable Eady solutions

Considering just the square root term, want to determine where the limit of instability in the solutions is by setting this part equal to zero. **This will define the short wave cutoff beyond which exponential growth will not occur.**

$$\frac{\alpha_c H^2}{4} - \alpha_c (\tanh \alpha_c H)^{-1} + 1 = 0$$

Employing the trigonometric identity

$$\tanh(\alpha_c H) = \frac{2\tanh(\alpha_c H/2)}{1+\tanh^2(\alpha_c H/2)}$$

Solution for short-wave cutoff:

$$\left[\frac{\alpha_c H}{2} - \tanh\left(\frac{\alpha_c H}{2}\right)\right]\left[\frac{\alpha_c H}{2} - \coth\left(\frac{\alpha_c H}{2}\right)\right] = 0$$

Solution for short-wave cutoff:

$$\left[\frac{\alpha_c H}{2} - \tanh\left(\frac{\alpha_c H}{2}\right)\right]\left[\frac{\alpha_c H}{2} - \coth\left(\frac{\alpha_c H}{2}\right)\right] = 0$$

The critical value for the shortwave cutoff:

$$\frac{\alpha_c H}{2} = \coth\left(\frac{\alpha_c H}{2}\right)$$

The approximate solution to this transcendental equation is:

$$\alpha_c H \approx 2.4$$

Or, if we assume square waves (k=l), the short wave cutoff considering the Rossby radius of deformation (L_r = NH/fo) is

$$k^2 + l^2 = \frac{5.76}{L_r}$$

The shortwave cutoff is the range of 2000-3000 km, depending on specific values of L_r.

Wavelength for maximum growth (L_m)

Considering k=l

$$L_m = 2\sqrt{2}\cdot\pi\cdot L_r \big/ H\alpha_m \approx 5500 km$$

→For $f_o = 10^{-4}\ s^{-1}$, H=10 km, $N^2 = 10^{-2}\ s^{-1}$

$\Lambda = 3 \times 10^{-3}\ s^{-1}$

α_m the value for which kc_i is a maximum

$$kc_{i\max} = \sigma_{\max} = 0.3098\,\frac{f_o}{N_o}\frac{\partial \overline{u}}{\partial z}$$

$$\frac{1}{kc_{i\max}} = T_e$$

E-folding timescale of most unstable wave (1-2 days)

$\sigma_{\max} \equiv 0.3098(f/N_*)\,dU/dz_*$.

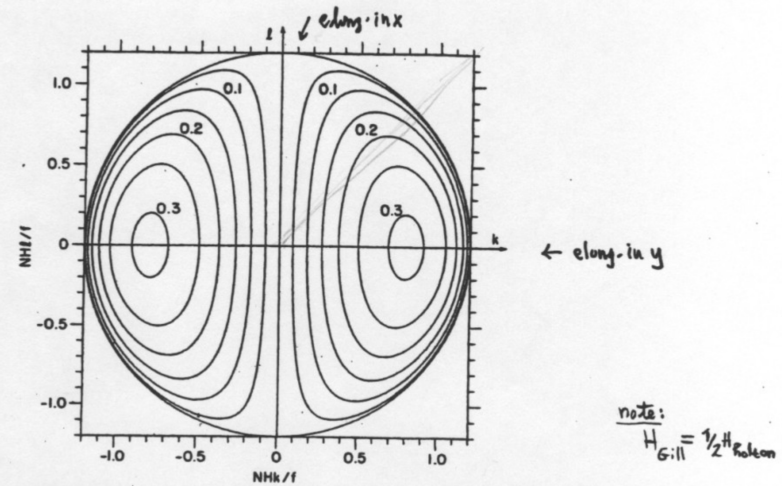


Fig. 13.3. Growth rate σ of an Eady wave as a function of wavenumber (k, l). Contours are shown in units of $(f/N_*\, dU/dz_*$. Values are zero on $k = 0$ and when the magnitude κ_H of the wavenumber equals 1.1997. The maximum value $0.3098(f/N_*)\,dU/dz_*$ is achieved when $l = 0$ and $N_* H\kappa_H/f = 0.8031$. For fixed ratio k/l, the maximum is at the same value of κ_H. The maximum for a fixed l (corresponding to a baroclinic zone of fixed width) is at a value of k that decreases as l increases (longer unstable waves for narrower baroclinic zones).

Characteristics of most unstable Eady mode

Tilted structure to highs and lows (as seen by streamfunction solution)

Isobars cross isotherms

Maximum southerly winds and rising motion ahead of upper-level low

Maximum northerly winds and sinking motion ahead of upper-level high

Just about what you get for a real mid-latitude cyclone in the mature to occluded stage!

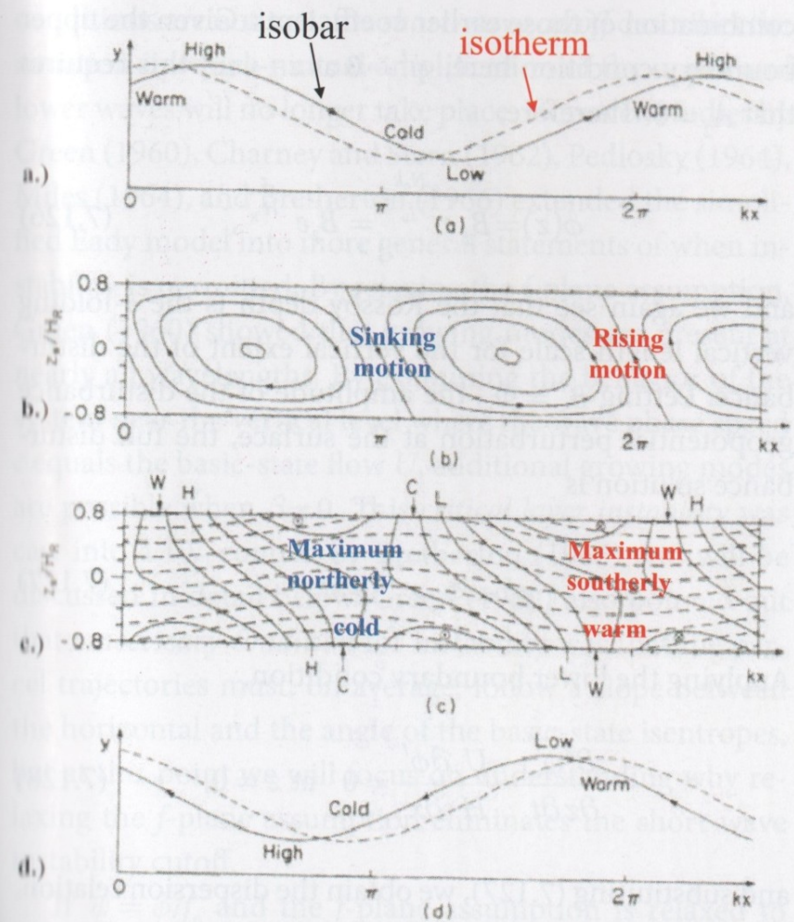


Figure 7.9. Structure of most unstable Eady wave: plan view of isobar and stream line at (a) the top of the domain ($z = H$) and (d) the lower boundary ($z = 0$), and cross section in x and z of (b) ageostrophic streamfunction, and (c) meridional wind (solid contours) and potential temperature (dashed contours; from Gill 1982).