Various Vertical Coordinate Systems Used for Numerical Weather Prediction

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ABSTRACT

For numerical weather prediction with primitive equations (the Eulerian hydrodynamic equations modified by the assumption of hydrostatic equilibrium), various coordinate systems are used to represent the vertical structure of the atmosphere. In this paper, we review the essential features of prediction equations, satisfying the conservation of mass and total energy, in various vertical coordinate systems. We formulate the equations of horizontal motion, hydrostatic balance, mass continuity, and thermodynamics using a generalized vertical coordinate in which any variable that gives a single-valued monotonic relationship with a geometric height can be used as a vertical coordinate. Conditions to conserve total energy in a generalized vertical coordinate are investigated.

Various prediction schemes using pressure, height, and potential temperature as a vertical coordinate are derived from the set of basic equations in the generalized coordinate system. These three coordinate systems are unique in that the features of prediction equations in each system are all distinct. We place special emphasis on handling the earth's orography as the lower boundary condition. As an extension of the original idea of Phillips applied to the pressure-coordinate system, we propose transformed height and isentropic systems. In those systems, both the top of the model atmosphere and the earth's surface are always coordinate surfaces. It is hoped that these new schemes, as in the case of the Phillips' sigma-system, will enable us to handle the effect of the earth's orography in the prediction models without lengthy coding logic.

1. Introduction

The motions of the atmosphere are governed by physical laws described by the equations of hydrodynamics and thermodynamics. These equations are well known in principle, although specification of frictional terms and heat sources as functions of dependent variables (state functions) in the equations requires knowledge of physical processes operating in the atmosphere. The evolution of flow patterns may be predicted by integrating the basic equations with respect to time starting from initial conditions. Since the basic equations are nonlinear and not amenable to solution by analytical methods, we must resort to the numerical approach. This requires the representation of dependent variables of the partial differential equations by a finite number of functions which depend on the independent variables—referred to as the coordinate variable.

The purpose of this paper is to review various vertical coordinate systems used for numerical weather prediction to represent the state functions in the vertical and to compare various prediction schemes based on primitive equations (the Eulerian hydrodynamic equations modified by the assumption of hydrostatic equilibrium). It is natural to consider geometrical altitude as a vertical coordinate as in Richardson (1922). Despite the conceptual simplicity of geometrical height, the height coordinate system had never been used extensively in numerical weather prediction until Kasahara and Washington (1967) reformulated Richardson's approach for high-speed computers.

The use of pressure as a vertical coordinate became very popular during the 1950s. As shown by Sutcliffe and Godart (Sutcliffe, 1947) and Eliassen (1949), the mass continuity equation reduces to a diagnostic equation in the pressure coordinate system and a measure of the vertical motion can be obtained in a simple form. This made the theoretical analysis of large-scale motions easier, particularly with quasi-geostrophic models. Many numerical models with pressure as a vertical coordinate were proposed and successfully tested. Among them were those of Charney and Phillips (1953) using a quasi-geostrophic model and Hinkelmann (1959) and Leith (1965) using primitive equation models.

The pressure coordinate system has certain computational disadvantages in the vicinity of the mountains because the lower limit of the atmosphere is not a coordinate surface. In fact, there have been very few attempts to incorporate the earth's orography in the pressure coordinate system. To circumvent this difficulty, Phillips (1957) proposed the so-called sigma-system, which is a modified version of the pressure

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coordinate system in that the earth's surface is always a coordinate surface. The sigma-coordinate system was adopted in many numerical weather prediction models in the 1960s based on primitive equations (e.g., Smagorinsky *et al.*, 1965).

Recently, interest has been revived in the use of potential temperature as a vertical coordinate for objective synoptic analysis (Shapiro and Hastings, 1973) and for weather prediction with a quasi-geostrophic model (Bleck, 1973) and with primitive equation models (Eliassen and Raustein, 1968, 1970; Shapiro, 1973). The use of potential temperature as a vertical coordinate seems to be particularly suitable for resolving details of frontal structure. Yet, in handling the lower boundary conditions in the isentropic coordinate system, we face the same degree of complexity as in the isobaric coordinate system.

It is still not clear which vertical coordinate system is best suited for numerical weather prediction, since each has its advantages and disadvantages. Major questions in choosing a vertical coordinate system are: Which system can represent the vertical structure of the atmosphere with the least amount of vertical resolution? Which system can best handle the earth's orography?

In the pressure, height and potential temperature coordinate systems, a special procedure is needed to take into account the effect of the earth's orography. This consists of examining in the computer program the height of mountains and dealing with lateral boundary conditions at the grid points in the vicinity of the mountains. This additional procedure slows down the calculations with advanced computers designed for processing arrays of data simultaneously.

Although the sigma-system, a modified version of the isobaric coordinate, is not free from shortcomings, as will be discussed later, the same idea of transforming the earth's surface to a coordinate surface is applicable to the height and potential temperature coordinate systems as well.

In deriving prediction schemes using different vertical coordinates, we find it convenient to introduce a general system that utilizes any well defined variable as a vertical coordinate. The idea of adopting a generalized vertical coordinate was stimulated by Starr (1945), who used a material surface in the vertical to formulate prediction equations. Since a Lagrangian coordinate is employed in the vertical, while Eulerian coordinates are chosen in the horizontal, Starr named the hybrid system a quasi-Lagrangian coordinate system. The generalized vertical coordinate discussed here need not be a Lagrangian coordinate. Rather, this is considered the transformation of a vertical coordinate into a more convenient form.

Two additional factors not explicitly discussed in the past are also examined in this review—derivation of the total energy equations in various systems and formulation of lower boundary conditions incorporating the earth's orography.

2. Basic atmospheric equations

We first summarize the basic atmospheric equations for large-scale flows using Cartesian coordinates x, y, and z directed eastward, northward, and upward.

The equation of horizontal motion may be expressed in the form[2]

$$\frac{d\mathbf{V}}{dt}+f\mathbf{k}\times\mathbf{V}=-\frac{1}{\rho}\nabla p+\mathbf{F} \tag{2.1}$$

where

$$\left.\begin{aligned}&\mathbf{V}=u\mathbf{i}+v\mathbf{j},\quad \nabla=\mathbf{i}\frac{\partial}{\partial x}+\mathbf{j}\frac{\partial}{\partial y}\\ &\frac{d}{dt}=\frac{\partial}{\partial t}+\mathbf{V}\cdot\nabla+w\frac{\partial}{\partial z}\end{aligned}\right\} \tag{2.1a}$$

in which $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ denote unit vectors in x-, y-, and z-coordinates, respectively; ∇ horizontal del operator, $\mathbf{V}$ horizontal velocity, u and v the x- and y-components of $\mathbf{V}$, w vertical velocity, d/dt total derivative, f Coriolis parameter ($\equiv 2\Omega\sin\varphi$), Ω angular velocity of the earth's rotation, φ geographical latitude, ρ density, p pressure and $\mathbf{F}$ frictional force per unit mass.

For large-scale motions, the hydrostatic equation

$$\frac{\partial p}{\partial z}=-\rho g, \tag{2.2}$$

where g denotes the earth's gravity, is a good approximation to the vertical equation of motion.

The mass continuity equation can be written as

$$\frac{d}{dt}\ln\rho+\nabla\cdot\mathbf{V}+\frac{\partial w}{\partial z}=0 \tag{2.3}$$

or

$$\frac{\partial \rho}{\partial t}+\nabla\cdot(\rho\mathbf{V})+\frac{\partial(\rho w)}{\partial z}=0. \tag{2.3a}$$

The first law of thermodynamics may be expressed by

$$\frac{d}{dt}\ln\theta=\frac{Q}{c_pT} \tag{2.4}$$

where θ is the potential temperature defined by

$$\theta\equiv T(p_0/p)^{\kappa} \tag{2.4a}$$

with $\kappa=R/c_p$ and $p_0=1013$ mb, and T is temperature given by the ideal gas law

$$p=\rho RT \tag{2.5}$$

[2] A list of symbols frequently used in the text appears in the Appendix.

in which R represents the specific gas constant. In (2.4), c_p stands for the specific heat at constant pressure and Q the rate of heating/cooling per unit mass per unit time. Here R and c_p have the following relation

$$R=c_p-c_v \tag{2.6}$$

where c_v denotes the specific heat at constant volume. Both c_p and c_v are assumed constant.

Another convenient form of the first law of thermodynamics is derived from (2.4) and (2.5) as

$$c_p\frac{dT}{dt}-\frac{1}{\rho}\frac{dp}{dt}=Q \tag{2.7}$$

or

$$\frac{d\rho}{dt}=\frac{1}{\gamma RT}\frac{dp}{dt}-\frac{\rho Q}{c_pT} \tag{2.8}$$

where $\gamma=c_p/c_v$.

The system of equations (2.1)–(2.5) constitutes the basic set of dynamical principles for numerical weather prediction. For prediction of large-scale weather phenomena, it is important to take into account water vapor in the atmosphere. We omit the prediction of the water vapor field in this paper since it is not directly connected with the choice of the vertical coordinate system. Thus, if the frictional term $\mathbf{F}$ and the heating term Q can be expressed by the dependent variables $\mathbf{V}$, w, p, and ρ (or T), the system, together with proper boundary conditions, forms a complete set.

Note that in (2.1) we introduce simplifications based on the so-called shallowness approximation (Phillips, 1966; Hinkelmann, 1969) to neglect minor terms. We also assume that the surface of constant apparent gravity potential is approximated by a sphere so that the geopotential depends only on the vertical coordinate z and the earth's gravity g is constant. We assume throughout that the scale factors for the two horizontal coordinates do not vary with height.

3. Generalized vertical coordinate system

Let the z-system be the coordinate system with independent variables x, y, z, t, and the s-system the generalized coordinate system with x, y, s, t, where s represents the generalized coordinate

$$s=s(x,y,z,t) \tag{3.1}$$

as a function of x, y, z, and t. We assume that the above equation gives a single-valued monotonic relationship between s and z, when x, y, and t are held fixed. Thus, by inverting (3.1) for z, it follows that

$$z=z(x,y,s,t). \tag{3.2}$$

Any scalar function A in the four-dimensional space may be expressed in either of two ways depending upon whether z or s is chosen as a vertical coordinate. Thus, the partial derivative of A with respect to c, where c can be x, y, or t, is generally different in the two systems and the following relationship exists:

$$\left(\frac{\partial A}{\partial c}\right)_s=\left(\frac{\partial A}{\partial c}\right)_z+\frac{\partial A}{\partial z}\left(\frac{\partial z}{\partial c}\right)_s \tag{3.3}$$

where the subscripts s and z indicate a particular vertical coordinate to be held constant for partial differentiation. Using the relationship

$$\frac{\partial A}{\partial z}=\left(\frac{\partial s}{\partial z}\right)\frac{\partial A}{\partial s}, \tag{3.4}$$

we can rewrite (3.3) as

$$\left(\frac{\partial A}{\partial c}\right)_s=\left(\frac{\partial A}{\partial c}\right)_z+\frac{\partial s}{\partial z}\left(\frac{\partial z}{\partial c}\right)_s\frac{\partial A}{\partial s}. \tag{3.5}$$

If we choose t for c, it follows that

$$\left(\frac{\partial A}{\partial t}\right)_s=\left(\frac{\partial A}{\partial t}\right)_z+\frac{\partial s}{\partial z}\left(\frac{\partial z}{\partial t}\right)_s\frac{\partial A}{\partial s}. \tag{3.6}$$

Similarly, choosing x and y for c, we obtain

$$\nabla_sA=\nabla_zA+\frac{\partial s}{\partial z}(\nabla_sz)\left(\frac{\partial A}{\partial s}\right). \tag{3.7}$$

We now transform the set of prediction equations in the z-system into the s-system. The total derivative d/dt in (2.1a) can be transformed to the s-system with the aid of (3.6) and (3.7). The result is

$$\frac{d}{dt}=\left(\frac{\partial}{\partial t}\right)_s+\mathbf{V}\cdot\nabla_s+\left[w-\left(\frac{\partial z}{\partial t}\right)_s-\mathbf{V}\cdot\nabla_sz\right]\frac{\partial s}{\partial z}\frac{\partial}{\partial s}. \tag{3.8}$$

Also, by the definition of total derivative in the s-system, we have

$$\frac{d}{dt}\equiv\left(\frac{\partial}{\partial t}\right)_s+\mathbf{V}\cdot\nabla_s+\dot{s}\frac{\partial}{\partial s}, \tag{3.9}$$

where $\dot{s}$ is the generalized vertical velocity

$$\dot{s}=ds/dt \tag{3.10}$$

which corresponds to w in the z-system

$$w\equiv dz/dt. \tag{3.11}$$

The relationship between $\dot{s}$ and w can be obtained by comparing (3.8) and (3.9). Thus,

$$\dot{s}=\frac{\partial s}{\partial z}\left[w-\left(\frac{\partial z}{\partial t}\right)_s-\mathbf{V}\cdot\nabla_sz\right]. \tag{3.12}$$

The horizontal equation of motion (2.1) can be transformed to the s-system as

$$\frac{d\mathbf{V}}{dt}+f\mathbf{k}\times\mathbf{V}=-\frac{1}{\rho}\nabla_s p+\frac{1}{\rho}\left(\frac{\partial s}{\partial z}\right)(\nabla_s z)\frac{\partial p}{\partial s}+\mathbf{F}. \quad (3.13)$$

By making use of the hydrostatic equation (2.2) and (3.4), we simplify the above as

$$\frac{d\mathbf{V}}{dt}+f\mathbf{k}\times\mathbf{V}=-\frac{1}{\rho}\nabla_s p-g\nabla_s z+\mathbf{F}. \quad (3.14)$$

This is the equation of horizontal motion in the hydrostatic s-system.

To transform the mass continuity equation (2.3) into the s-system, note that from (3.12) we have

$$w=\left(\frac{\partial z}{\partial t}\right)_s+\mathbf{V}\cdot\nabla_s z+\dot{s}\frac{\partial z}{\partial s}. \quad (3.15)$$

Thus,

$$\frac{\partial w}{\partial z}=\frac{\partial w}{\partial s}\frac{\partial s}{\partial z}$$

$$=\frac{\partial s}{\partial z}\left[\frac{d}{dt}\left(\frac{\partial z}{\partial s}\right)+\frac{\partial \mathbf{V}}{\partial s}\cdot\nabla_s z\right]+\frac{\partial \dot{s}}{\partial s}. \quad (3.16)$$

Also, with the aid of (3.5), we can show that

$$\nabla_z\cdot\mathbf{V}=\nabla_s\cdot\mathbf{V}-\left(\frac{\partial s}{\partial z}\right)(\nabla_s z)\cdot\frac{\partial \mathbf{V}}{\partial s}. \quad (3.17)$$

Substitution of (3.16) and (3.17) into (2.3) yields the mass continuity equation in the s-system:

$$\frac{d}{dt}\ln\left(\rho\frac{\partial z}{\partial s}\right)+\nabla_s\cdot\mathbf{V}+\frac{\partial \dot{s}}{\partial s}=0 \quad (3.18)$$

which can be rewritten as

$$\left[\frac{\partial}{\partial t}\left(\rho\frac{\partial z}{\partial s}\right)\right]_s+\nabla_s\cdot\left(\rho\mathbf{V}\frac{\partial z}{\partial s}\right)+\frac{\partial}{\partial s}\left(\rho\dot{s}\frac{\partial z}{\partial s}\right)=0. \quad (3.19)$$

The hydrostatic equation (2.2) in the s-system can be expressed, using (3.4), as

$$\rho\frac{\partial z}{\partial s}=-\frac{1}{g}\frac{\partial p}{\partial s}. \quad (3.20)$$

With the aid of the above equation, we may rewrite (3.19) as

$$\frac{\partial}{\partial s}\left(\frac{\partial p}{\partial t}\right)_s+\nabla_s\cdot\left(\mathbf{V}\frac{\partial p}{\partial s}\right)+\frac{\partial}{\partial s}\left(\dot{s}\frac{\partial p}{\partial s}\right)=0. \quad (3.21)$$

The hydrostatic equilibrium (2.2) or (3.20) is assumed throughout the remainder of this article.

Since the thermodynamic equation (2.7) or (2.8) is written in terms of the total derivatives of dependent variables, the thermodynamic equation in the s-system has the same form as (2.7) or (2.8) except that the total derivative in the s-system is defined by (3.9).

4. Upper and lower boundary conditions

In predicting the atmospheric flow with a numerical model, we cannot handle a vertical coordinate which literally extends to infinity. Thus, it is necessary to set an upper boundary of the model and corresponding boundary conditions as required by a solution to the problem. The requirement may be different depending on the specific vertical coordinate to be chosen and the characteristics of prediction equations.

With regard to the kinematic condition, it is natural and convenient to select the upper boundary as a vertical coordinate surface $s=s_T=\text{const}$. We then assume that there is no mass transport through this surface. This is stated as

$$\dot{s}=0 \quad \text{at} \quad s=s_T=\text{const}. \quad (4.1)$$

As a lower boundary condition of the atmosphere, we assume that there is no mass transport through the earth's surface, which is located at fixed altitude H above the mean sea level $z=0$. In the s-system, the earth's surface is expressed by

$$s=s_H\equiv s(x,y,H,t). \quad (4.2)$$

The value of s at $z=H$ may vary with time and space. As a direct consequence of the fact that air at the earth's surface may move only along the earth's surface, we get the lower boundary condition

$$\dot{s}=\frac{\partial s_H}{\partial t}+\mathbf{V}_H\cdot\nabla s_H \quad \text{at} \quad s=s_H \quad (4.3)$$

where $\mathbf{V}_H$ denotes the horizontal velocity at $z=H$.

Now, by integrating the continuity equation (3.19) with respect to s from s_H to s_T, exchanging the order of integrations and differentiations (remembering that s_H is a function of x, y, and t) and considering the upper and lower boundary conditions (4.1) and (4.3), we obtain

$$\frac{\partial}{\partial t}\int_{s_H}^{s_T}\rho\left(\frac{\partial z}{\partial s}\right)ds+\nabla\cdot\int_{s_H}^{s_T}\mathbf{V}\rho\left(\frac{\partial z}{\partial s}\right)ds=0. \quad (4.4)$$

When integrated horizontally, the above gives the statement of mass conservation—the mass contained between s_T and s_H is constant. In fact, the boundary conditions are so chosen to conserve the total mass in the prediction model.

If the earth's surface coincides with a constant s-surface, then the lower boundary condition (4.3) reduces to

$$\dot{s}=0 \quad \text{at} \quad s=s_H, \quad (4.5)$$

which will be applied in transformed coordinate systems discussed later.

5. Energy equations in the s-coordinate system

Scalar multiplication of (3.14) by $\mathbf{V}$ yields the kinetic energy equation

$$\left(\frac{\partial k}{\partial t}\right)_s+\mathbf{V}\cdot\nabla_s k+\dot{s}\frac{\partial k}{\partial s}=-\frac{1}{\rho}\mathbf{V}\cdot\nabla_s p-g\mathbf{V}\cdot\nabla_s z+\mathbf{V}\cdot\mathbf{F}, \quad (5.1)$$

where

$$k\equiv\tfrac{1}{2}\mathbf{V}\cdot\mathbf{V}=\tfrac{1}{2}(u^2+v^2) \quad (5.1a)$$

is the horizontal kinetic energy per unit mass.

Multiplying (5.1) and (3.19) by $\rho\partial z/\partial s$ and k, respectively, and adding the resulting two equations, we obtain

$$\left[\frac{\partial}{\partial t}\left(k\rho\frac{\partial z}{\partial s}\right)\right]_s+\nabla_s\cdot\left(k\mathbf{V}\rho\frac{\partial z}{\partial s}\right)+\frac{\partial}{\partial s}\left(k\dot{s}\rho\frac{\partial z}{\partial s}\right)$$
$$=-\frac{\partial z}{\partial s}\mathbf{V}\cdot\nabla_s p-g\rho\frac{\partial z}{\partial s}\mathbf{V}\cdot\nabla_s z+\rho\frac{\partial z}{\partial s}\mathbf{V}\cdot\mathbf{F}. \quad (5.2)$$

Multiplying (2.7) by $\rho\partial z/\partial s$, expanding dT/dt and dp/dt and using the continuity equation (3.19), we find

$$\left[\frac{\partial}{\partial t}\left(c_pT\rho\frac{\partial z}{\partial s}\right)\right]_s+\nabla_s\cdot\left(c_pT\mathbf{V}\rho\frac{\partial z}{\partial s}\right)+\frac{\partial}{\partial s}\left(c_pT\dot{s}\rho\frac{\partial z}{\partial s}\right)$$
$$-\frac{\partial z}{\partial s}\left[\left(\frac{\partial p}{\partial t}\right)_s+\mathbf{V}\cdot\nabla_s p+\dot{s}\frac{\partial p}{\partial s}\right]=\rho\frac{\partial z}{\partial s}Q. \quad (5.3)$$

Adding (5.2) and (5.3), it follows that

$$\left\{\frac{\partial}{\partial t}\left[(k+c_pT)\rho\frac{\partial z}{\partial s}\right]\right\}_s$$
$$+\nabla_s\cdot\left[(k+c_pT)\mathbf{V}\rho\frac{\partial z}{\partial s}\right]+\frac{\partial}{\partial s}\left[(k+c_pT)\dot{s}\rho\frac{\partial z}{\partial s}\right]$$
$$=\frac{\partial z}{\partial s}\left(\frac{\partial p}{\partial t}\right)_s+\frac{\partial z}{\partial s}\dot{s}\frac{\partial p}{\partial s}-g\rho\frac{\partial z}{\partial s}\mathbf{V}\cdot\nabla_s z+\rho\frac{\partial z}{\partial s}(Q+\mathbf{V}\cdot\mathbf{F}). \quad (5.4)$$

Multiplying the hydrostatic equation (3.20) by $gz\dot{s}$ and differentiating with respect to s, we have

$$\frac{\partial z}{\partial s}\dot{s}\frac{\partial p}{\partial s}=-\frac{\partial}{\partial s}\left(gz\dot{s}\rho\frac{\partial z}{\partial s}\right)-z\frac{\partial}{\partial s}\left(\dot{s}\frac{\partial p}{\partial s}\right). \quad (5.5)$$

Operating ∇_s to $gz\mathbf{V}\rho(\partial z/\partial s)$ and substituting the hydrostatic equation, we find

$$-g\rho\frac{\partial z}{\partial s}\mathbf{V}\cdot\nabla_s z=-\nabla_s\cdot\left(gz\mathbf{V}\rho\frac{\partial z}{\partial s}\right)-z\nabla_s\cdot\left(\mathbf{V}\frac{\partial p}{\partial s}\right). \quad (5.6)$$

Using (5.5), (5.6) and the continuity equation (3.21) multiplied by z, (5.4) is expressed by

$$\left\{\frac{\partial}{\partial t}\left[(k+c_pT)\rho\frac{\partial z}{\partial s}\right]\right\}_s+\nabla_s\cdot\left[(k+c_pT+gz)\mathbf{V}\rho\frac{\partial z}{\partial s}\right]$$
$$+\frac{\partial}{\partial s}\left[(k+c_pT+gz)\dot{s}\rho\frac{\partial z}{\partial s}\right]$$
$$=\frac{\partial}{\partial s}\left(z\frac{\partial p}{\partial t}\right)+\rho\frac{\partial z}{\partial s}(Q+\mathbf{V}\cdot\mathbf{F}). \quad (5.7)$$

This is the total energy equation in the hydrostatic s-system.

Let us integrate (5.7) with respect to s from s_H to s_T. By exchanging the order of integrations and differentiations (remembering that s_H is a function of x, y, and t) and applying the upper and lower boundary conditions (4.1) and (4.3), we obtain

$$\frac{\partial}{\partial t}\int_{s_H}^{s_T}(k+c_pT)\rho\frac{\partial z}{\partial s}ds+\nabla\cdot\int_{s_H}^{s_T}(k+c_pT+gz)\mathbf{V}\rho\frac{\partial z}{\partial s}ds$$
$$=gH\left(\rho\frac{\partial z}{\partial s}\right)_H\frac{\partial s_H}{\partial t}+\left[z\left(\frac{\partial p}{\partial t}\right)_s\right]_{s=s_T}-\left[z\left(\frac{\partial p}{\partial t}\right)_s\right]_{s=s_H}$$
$$+\int_{s_H}^{s_T}(Q+\mathbf{V}\cdot\mathbf{F})\rho\frac{\partial z}{\partial s}ds. \quad (5.8)$$

We now evaluate the second and third terms on the right of (5.8). Since $s_T=$const., the second term becomes

$$\left[z\left(\frac{\partial p}{\partial t}\right)_s\right]_{s=s_T}=z_T\frac{\partial p_T}{\partial t}, \quad (5.9)$$

where z_T and p_T denote height and pressure at $s=s_T$. Although s_T is constant, z_T can be a function of x, y, and t in the s-system except in a height coordinate where z_T is constant. The value of $\partial p_T/\partial t$ exists in the s-system except in a pressure coordinate where $\partial p_T/\partial t$ vanishes.

Observing that s_H is a function of time in general, we find

$$\left[\left(\frac{\partial p}{\partial t}\right)_s\right]_{s=s_H}=\frac{\partial p_H}{\partial t}-\left(\frac{\partial p}{\partial s}\right)_{s_H}\frac{\partial s_H}{\partial t}$$
$$=\frac{\partial p_H}{\partial t}+g\left(\rho\frac{\partial z}{\partial s}\right)_H\frac{\partial s_H}{\partial t}. \quad (5.10)$$

The hydrostatic equation is used in the last step. Note that $\partial p/\partial t$ should vanish on a pressure surface when s is replaced with p.

By substituting (5.10), multiplied by H, and (5.9) into (5.8), we have

$$\frac{\partial}{\partial t}\int_{sH}^{sT}(k+c_pT)\rho\frac{\partial z}{\partial s}ds+\nabla\cdot\int_{sH}^{sT}(k+c_pT+gz)\mathbf{V}\rho\frac{\partial z}{\partial s}ds$$
$$=z_T\frac{\partial p_T}{\partial t}-H\frac{\partial p_H}{\partial t}+\int_{sH}^{sT}(Q+\mathbf{V}\cdot\mathbf{F})\rho\frac{\partial z}{\partial s}ds. \quad (5.11)$$

This is the vertically integrated total energy equation in the hydrostatic s-system.

a. Geometrical height coordinate case

When height z is used as a vertical coordinate, the upper and lower boundary conditions corresponding to (4.1) and (4.3) are written, respectively, as

$$\left.\begin{aligned} &w=0 && \text{at } z=z_T=\text{const.}\\ &w=w_H=\mathbf{V}_H\cdot\nabla H && \text{at } z=H \end{aligned}\right\} \quad (5.12)$$

remembering that the height of mountains does not change with time.

The vertically integrated total energy equation in the z-system is obtained by replacing s with z in (5.11):

$$\frac{\partial}{\partial t}\int_{H}^{zT}(k+c_pT)\rho dz+\nabla\cdot\int_{H}^{zT}(k+c_pT+gz)\mathbf{V}\rho dz$$
$$=z_T\frac{\partial p_T}{\partial t}-H\frac{\partial p_H}{\partial t}+\int_{H}^{zT}(Q+\mathbf{V}\cdot\mathbf{F})\rho dz. \quad (5.13)$$

By integrating p with respect to z from H to z_T, using the equation of state (2.5) and definition of R by (2.6), we can show, as did Haurwitz (1941, p. 241), that

$$\int_{H}^{zT}c_pT\rho dz=\int_{H}^{zT}(c_vT+gz)\rho dz+z_Tp_T-Hp_H. \quad (5.14)$$

Differentiating the above with time, observing that z_T and H are independent of time and substituting the resulting equation into (5.13), we see that

$$\frac{\partial}{\partial t}\int_{H}^{zT}(k+c_vT+gz)\rho dz+\nabla\cdot\int_{H}^{zT}\mathbf{V}(k+c_pT+gz)\rho dz$$
$$=\int_{H}^{zT}(Q+\mathbf{V}\cdot\mathbf{F})\rho dz. \quad (5.15)$$

This is the vertically integrated total energy equation in the hydrostatic z-system expressed in conservation form. The quantities ρk, $c_v\rho T$, and $g\rho z$, respectively, represent the kinetic, internal, and potential energy per unit volume (van Mieghem, 1973).

b. Pressure coordinate case

When pressure p is used as the vertical coordinate, the upper and lower boundary conditions corresponding to (4.1) and (4.3) are expressed, respectively, as

$$\left.\begin{aligned} &\omega=\frac{dp}{dt}=0 && \text{at } p=p_T=\text{const.}\\ &\omega=\omega_H=\frac{\partial p_H}{\partial t}+\mathbf{V}_H\cdot\nabla p_H && \text{at } p=p_H \end{aligned}\right\} \quad (5.16)$$

where p_T and p_H denote pressure at the model top, which is constant, and at the earth's surface, which is variable in time and space.

The vertically integrated total energy equation in the p-system is obtained by replacing s with p in (5.11), remembering that p_T is a coordinate surface and using the hydrostatic equation,

$$\frac{\partial}{\partial t}\int_{pT}^{pH}(k+c_pT)dp+\nabla\cdot\int_{pT}^{pH}(k+c_pT+gz)\mathbf{V}dp$$
$$=-gH\frac{\partial p_H}{\partial t}+\int_{pT}^{pH}(Q+\mathbf{V}\cdot\mathbf{F})dp. \quad (5.17)$$

Since H does not depend on time, the above equation can be put in the conservation form

$$\frac{\partial}{\partial t}\int_{pT}^{pH}(k+c_pT+gH)dp+\nabla\cdot\int_{pT}^{pH}(k+c_pT+gz)\mathbf{V}dp$$
$$=\int_{pT}^{pH}(Q+\mathbf{V}\cdot\mathbf{F})dp. \quad (5.18)$$

This is the vertically integrated total energy equation in the p-system as given in Phillips (1973) for $p_T=0$.

c. General case—other than z- and p-systems

As far as the vertically integrated total energy equation, the z- and p-systems are unique in that the upper and lower boundary conditions (4.1) and (4.3), when properly expressed in p or z, are sufficient to cast the energy equation into conservation law form.

In the potential temperature coordinate discussed later, we need an additional upper boundary condition to put the vertically integrated total energy equation in conservation law form. We have two choices:

$$\text{(a)}\quad \partial p_T/\partial t=0 \quad \text{at } \theta=\theta_T=\text{const.} \quad (5.19)$$

$$\text{(b)}\quad z_T=\text{const.} \quad \text{for } \theta=\theta_T=\text{const.} \quad (5.20)$$

where θ_T is a constant potential temperature of the model top.

In the case of (a), the vertically integrated total energy equation in the θ-system can be expressed like that of the p-system. In (b), we must calculate $\partial p_T/\partial t$

on $\theta=\theta_T$ and the vertically integrated total energy equation becomes identical to that in the z-system. It is inconsistent, in general, to assume conditions (a) and (b) at the same time.

6. Prediction equations in pressure coordinates

We take pressure p as a vertical coordinate and define the individual change of p as

$$\omega \equiv dp/dt \tag{6.1}$$

where, using (3.9), the total derivative becomes

$$\frac{d}{dt}=\left(\frac{\partial}{\partial t}\right)_p+\mathbf{V}\cdot\nabla_p+\omega\frac{\partial}{\partial p}. \tag{6.2}$$

Since the first term on the right of (3.14) vanishes on isobaric surfaces, the horizontal equation of motion (3.14) reduces to

$$\frac{d\mathbf{V}}{dt}+f\mathbf{k}\times\mathbf{V}=-g\nabla_p z+\mathbf{F}. \tag{6.3}$$

After substituting p for s in (3.18) and (3.20), we find that the first term of (3.18) becomes $d\ln(-g^{-1})/dt$ which is identically zero. Thus, the continuity equation in the p-system takes a simple diagnostic form

$$\nabla_p\cdot\mathbf{V}+\frac{\partial\omega}{\partial p}=0. \tag{6.4}$$

By integrating the above equation with respect to p and using the upper boundary condition in (5.16), we obtain

$$\omega=-\int_{p_T}^{p}\nabla_p\cdot\mathbf{V}dp. \tag{6.5}$$

If we extend the integration to the earth's surface where the surface pressure is denoted by p_H, we have

$$\omega_H=-\int_{p_T}^{p_H}\nabla_p\cdot\mathbf{V}dp \tag{6.6}$$

where ω_H denotes the value of ω at the earth's surface $z=H$.

Elimination of ρ between the hydrostatic equation (2.2) and the equation of state (2.5) yields

$$g\frac{\partial z}{\partial p}=-\frac{RT}{p}. \tag{6.7}$$

By integrating the above with respect to p from the surface pressure p_H at $z=H$, we find

$$gz=gH+R\int_p^{p_H}Td(\ln p). \tag{6.8}$$

The thermodynamic equation (2.4) in the p-system may be expressed by

$$\left(\frac{\partial T}{\partial t}\right)_p+\mathbf{V}\cdot\nabla_p T+\frac{\omega T}{\theta}\frac{\partial\theta}{\partial p}=\frac{Q}{c_p}. \tag{6.9}$$

The surface pressure p_H is predicted from the surface pressure tendency equation

$$\frac{\partial p_H}{\partial t}=-\mathbf{V}_H\cdot\nabla p_H-\int_{p_T}^{p_H}\nabla_p\cdot\mathbf{V}dp, \tag{6.10}$$

which can be derived from the lower boundary condition in (5.16) and expression of ω_H by (6.6).

The initial conditions require the fields of $\mathbf{V}$, T, and p_H. Prognostic equations (6.3), (6.9), and (6.10) are used to forecast $\mathbf{V}$, T, and p_H. Diagnostic equations (6.5) and (6.8) are used to calculate ω and z.

Although the prediction equations in the pressure coordinate system are simple in form, it is awkward to handle the lower boundary condition since the surface pressure can vary with time and space. This produces the coding problem mentioned in the Introduction. Phillips (1957) introduced the so-called sigma-coordinate system, which is a transformed pressure coordinate mapping the earth's surface onto a coordinate surface. The sigma-coordinate system has been adopted in many primitive equation models (e.g., Holloway and Manabe, 1971). A variation of the sigma-coordinate has been used in general circulation models (e.g., Arakawa, 1972) and short-range prediction models (e.g., Shuman and Hovermale, 1968).

We define the σ-system as

$$\left.\begin{aligned}\sigma&=(p-p_T)/P_*\\ \text{and}\quad P_*&=p_H-p_T\end{aligned}\right\} \tag{6.11}$$

where p_H denotes the surface pressure at $z=H$ and p_T is a constant pressure corresponding to the model top. Thus, the model top is expressed by $\sigma=0$ and the earth's surface by $\sigma=1$, where no special checking is required to deal with the orography. If the model top is extended to $p_T\rightarrow 0$, (6.11) is identical to the one proposed by Phillips (1957).

Since $\nabla_\sigma\sigma$ vanishes on a σ-surface, by expanding terms we find

$$\nabla_\sigma\sigma=\frac{1}{P_*}(\nabla_\sigma p-\sigma\nabla P_*)=0.$$

Note that the del operator to P_* should be calculated on the earth's surface. Thus, the horizontal pressure gradient force on σ-surfaces is expressed, with the aid of the equation of state (2.5), by

$$\frac{1}{\rho}\nabla_\sigma p=\frac{\sigma RT}{\sigma P_*+p_T}\nabla P_*.$$

Hence, we write the horizontal equation of motion (3.14) as

$$\frac{d\mathbf{V}}{dt}+f\mathbf{k}\times\mathbf{V}=-g\nabla_\sigma z-\frac{\sigma RT}{\sigma P_*+p_T}\nabla P_*+\mathbf{F}. \quad (6.12)$$

The hydrostatic equation (3.20) can be put in the form

$$g\frac{\partial z}{\partial\sigma}=-\frac{RTP_*}{\sigma P_*+p_T}. \quad (6.13)$$

By integrating the above with respect to σ from the earth's surface $\sigma=1$, we obtain the equation of geopotential

$$gz=gH+RP_*\int_\sigma^1 \frac{T}{\sigma P_*+p_T}d\sigma. \quad (6.14)$$

Applying $\partial p/\partial\sigma=P_*$ in (3.21), we obtain the continuity equation in the σ-system,

$$\frac{\partial P_*}{\partial t}+\nabla_\sigma\cdot(P_*\mathbf{V})+P_*\frac{\partial\dot\sigma}{\partial\sigma}=0 \quad (6.15)$$

remembering that P_* is not a function of σ.

Corresponding to (4.1) and (4.5), we choose as the upper and lower boundary conditions, respectively:

$$\left.\begin{aligned}\dot\sigma=0 \quad \text{at } \sigma=0,\ p=p_T\\ \dot\sigma=0 \quad \text{at } \sigma=1,\ p=p_H\end{aligned}\right\}. \quad (6.16)$$

By integrating (6.15) with respect to σ and using the upper boundary condition (6.16), we get

$$\sigma\frac{\partial P_*}{\partial t}+\int_0^\sigma \nabla_\sigma\cdot(P_*\mathbf{V})d\sigma+P_*\dot\sigma=0. \quad (6.17)$$

The extension of the limit of the integration (6.17) to the earth's surface, with the lower boundary condition in (6.16), gives

$$\frac{\partial P_*}{\partial t}=-\int_0^1 \nabla_\sigma\cdot(P_*\mathbf{V})d\sigma. \quad (6.18)$$

By substituting the above equation back into (6.17), we obtain

$$\dot\sigma=-\frac{1}{P_*}\int_0^\sigma \nabla_\sigma\cdot(P_*\mathbf{V})d\sigma+\frac{\sigma}{P_*}\int_0^1 \nabla_\sigma\cdot(P_*\mathbf{V})d\sigma. \quad (6.19)$$

From definition (6.11), the individual change of σ is

$$\dot\sigma=\frac{1}{P_*}\left(\frac{dp}{dt}-\sigma\frac{dP_*}{dt}\right). \quad (6.20)$$

The individual change of P_* is given by the lower boundary condition in (5.16) by replacing p_H with P_*:

$$\frac{dP_*}{dt}=\frac{\partial P_*}{\partial t}+\mathbf{V}_H\cdot\nabla P_*. \quad (6.21)$$

By applying (6.20), (6.21), (6.17), and (6.18), we obtain the individual change of pressure,

$$\frac{dp}{dt}=\sigma\mathbf{V}_H\cdot\nabla P_*-\int_0^\sigma \nabla_\sigma\cdot(P_*\mathbf{V})d\sigma. \quad (6.22)$$

The thermodynamic equation in the σ-system may be obtained from (2.7), with the aid of the equation of state, as

$$\frac{dT}{dt}=\frac{RT}{c_p(P_*\sigma+p_T)}\frac{dp}{dt}+\frac{Q}{c_p} \quad (6.23)$$

in which dp/dt is already given by (6.22).

Prognostic equations (6.12), (6.23), and (6.18) are used to predict the fields of $\mathbf{V}$, T, and P_*, respectively. Diagnostic equations (6.19) and (6.14) calculate $\dot\sigma$ and z.

The vertically integrated total energy equation in the σ-system can be derived from (5.11) by replacing s with σ. Since $\partial p_T/\partial t=0$, we obtain the total energy equation identical to (5.18) after changing the integration variable and the integration limits. Comparison of the total energy equations in the p- and σ-systems is made by Haltiner (1971) by deriving the energy equations separately in the two systems, but the lower boundary condition was not considered rigorously in his treatment, as pointed out by Shuman (1973).

We should mention one computational problem related to the application of the σ-system over steep-slope mountains. As seen from (6.12), the horizontal pressure gradient force is calculated as the sum of two terms. In the case of p_T being zero, one term is the geopotential derivative on a sloping σ-surface $g\nabla_\sigma z$ and the other is the hydrostatic correction $RT\nabla(\ln p_H)$. The hydrostatic component must cancel out between the two terms to yield a proper evaluation of the pressure gradient in the σ-system. As pointed out by Gary (1973), this process can introduce considerable error appearing not only near the mountains, but also extending throughout the entire model atmosphere. This difficulty has been recognized for some time and various techniques have been suggested to overcome it. Smagorinsky *et al.* (1967) and Kurihara (1968) calculate the pressure gradient on pressure surfaces by vertically interpolating data from the σ-system to the p-system. Corby *et al.* (1972) proposed a special finite-difference scheme to evaluate the hydrostatic correction term $RT\nabla(\ln p_H)$ so that the hydrostatic components in the two terms of the pressure gradient in the σ-system cancel identically when there is no gradient of geopotential on isobaric surfaces and the temperature varies as $\log p$. Gary (1973) and Phillips (1973) suggest a scheme to remove the hydrostatic component from each of the two terms $g\nabla_\sigma z$ and $RT\nabla(\ln p_H)$ by assuming an average pressure profile in the neighborhood of the grid point where the pressure gradient is calculated. Although the above techniques offer considerable improvement in idealized situations, the same degree of improvement may not be attained in more realistic situations in model calculations with coarse horizontal and vertical resolutions. Further research is needed for improved application of the σ-system to numerical prediction.

7. Prediction equations in the height coordinate system

In the height coordinate system of Section 2, we presented three time-dependent equations—(2.1), (2.3) and (2.8)—for $\mathbf{V}$, ρ and p, respectively. The two diagnostic equations (2.2) and (2.5) are supposedly for w and T. However, it is not obvious how vertical velocity w can be computed. In the hydrostatic system, the calculations of $\partial\rho/\partial t$ and $\partial p/\partial t$ must satisfy (2.2). Since $\partial\rho/\partial t$ may be calculated from (2.3) and $\partial p/\partial t$ from (2.8), where both equations contain w, the value of w should be determined to satisfy the hydrostatic constraint. This diagnostic procedure to determine w was originally proposed by Richardson (1922) for the atmosphere of infinite height. In the p- and σ-systems, the vertical motion field is determined kinematically from the vertical profile of horizontal mass divergence, whereas in the z-system the vertical distribution of heating rate is required in addition to the vertical distributions of pressure and horizontal mass divergence. Such an equation of vertical motion is called the *Richardson equation.*

Kasahara and Washington (1967) modified Richardson's original formulation to the atmosphere of a finite depth. One significant difference in the atmosphere of a finite depth is the need for calculating the pressure tendency $\partial p_T/\partial t$ at the model top $z=z_T$. Although the form of equation for the vertical motion and the top pressure tendency appears complicated, there is no difficulty in executing the calculations.

Later, Kasahara and Washington (1971) presented a computing procedure to deal with the earth's orography by blocking the integration domain covered by mountains. This requires examining in the computer program the height of mountains and dealing with lateral boundary conditions at the grid points in the vicinity of mountains. One advantage of this procedure is that the truncation error is confined mostly to the mountains rather than extended throughout the model atmosphere as in the case of the σ-system.

The coding of the program is actually straightforward for a second-order finite-difference scheme, but it may become complicated for higher-order, finite-difference schemes and expensive in terms of additional computer time, particularly on advanced computers designed to process arrays of data simultaneously. In search of a more effective scheme to handle the earth's orography in the z-system to fully utilize future computers, we propose here prediction equations in transformed height coordinates in which the earth's surface and the top of the model atmosphere are both coordinate surfaces.

For example, consider a coordinate $\tilde{z}$ defined by

$$\tilde{z}=\frac{z_T-z}{z_T-H} \qquad (7.1)$$

where z_T denote a constant height of the model top. Since $\tilde{z}=0$ at $z=z_T$ and $\tilde{z}=1$ at $z=H$, the coordinate (7.1) satisfies the requirement that both the earth's surface and the model top be coordinate surfaces. Thus, (7.1) becomes similar to the sigma coordinate in Section 6 with one important difference in (7.1)—that at constant z, $\tilde{z}$ can be a function of x and y, but not of time.

In this section, we present the set of prediction equations in a more general transformed height coordinate ζ, *by assuming only that at constant height z, ζ is independent of time and that both the model top and the earth's surface are constant ζ surfaces.*

The horizontal equation of motion in this system is obtained from (3.14) by replacing s with ζ and using (2.5):

$$\frac{d\mathbf{V}}{dt}+f\mathbf{k}\times\mathbf{V}=-g\nabla_\zeta z-RT\nabla_\zeta(\ln p)+\mathbf{F}, \qquad (7.2)$$

where

$$\frac{d}{dt}=\left(\frac{\partial}{\partial t}\right)_\zeta+\mathbf{V}\cdot\nabla_\zeta+\dot{\zeta}\frac{\partial}{\partial\zeta} \qquad (7.3)$$

and

$$\dot{\zeta}=d\zeta/dt. \qquad (7.4)$$

The hydrostatic equation (3.20) becomes

$$\frac{\partial p}{\partial\zeta}=-g\rho\frac{\partial z}{\partial\zeta}. \qquad (7.5)$$

The continuity equation is obtained from (3.21) by replacing s with ζ:

$$\frac{\partial}{\partial\zeta}\left(\frac{\partial p}{\partial t}\right)_\zeta+\nabla_\zeta\cdot\left(\mathbf{V}\frac{\partial p}{\partial\zeta}\right)+\frac{\partial}{\partial\zeta}\left(\dot{\zeta}\frac{\partial p}{\partial\zeta}\right)=0. \qquad (7.6)$$

By integrating the above equation with respect to ζ from some level ζ to the model top ζ_T and applying the upper boundary condition,

$$\dot{\zeta}=0 \quad \text{at} \quad \zeta=\zeta_T, \qquad (7.7)$$

we obtain the pressure tendency equation:

$$\left(\frac{\partial p}{\partial t}\right)_\zeta=\frac{\partial p_T}{\partial t}-\dot{\zeta}\frac{\partial p}{\partial \zeta}+\int_\zeta^{\zeta_T}\nabla_\zeta\cdot\left(\mathbf{V}\frac{\partial p}{\partial \zeta}\right)d\zeta, \tag{7.8}$$

where $\partial p_T/\partial t$ denotes the pressure tendency at the model top.

By substituting the above into the definition of dp/dt in the system, we find

$$\frac{dp}{dt}=\frac{\partial p_T}{\partial t}+J \tag{7.9}$$

where

$$J=\mathbf{V}\cdot\nabla_\zeta p+\int_\zeta^{\zeta_T}\nabla_\zeta\cdot\left(\mathbf{V}\frac{\partial p}{\partial \zeta}\right)d\zeta. \tag{7.10}$$

Now we discuss how to obtain the vertical velocity $\dot{\zeta}$ and the top pressure tendency $\partial p_T/\partial t$. By replacing s with ζ in (3.18) and noting at constant height z that $\partial z/\partial\zeta$ is independent of time, we obtain the continuity equation in the form

$$\frac{\partial z}{\partial \zeta}\frac{1}{\rho}\frac{d\rho}{dt}=-\nabla_\zeta\cdot\left(\mathbf{V}\frac{\partial z}{\partial \zeta}\right)-\frac{\partial}{\partial \zeta}\left(\dot{\zeta}\frac{\partial z}{\partial \zeta}\right). \tag{7.11}$$

Multiplying (2.8) by $\rho^{-1}\partial z/\partial\zeta$ and substituting (7.11) and (7.9) into the resulting equation, we get

$$\frac{\partial}{\partial \zeta}\left(\dot{\zeta}\frac{\partial z}{\partial \zeta}\right)=-\nabla_\zeta\cdot\left(\mathbf{V}\frac{\partial z}{\partial \zeta}\right)-\frac{\dfrac{\partial p_T}{\partial t}+J}{\gamma p}\frac{\partial z}{\partial \zeta}+\frac{Q}{c_pT}\frac{\partial z}{\partial \zeta}. \tag{7.12}$$

By integrating the above equation with respect to ζ from the earth's surface ζ_H and applying the lower boundary condition

$$\dot{\zeta}=0 \quad\text{at}\quad \zeta=\zeta_H, \tag{7.13}$$

we obtain the expression for $\dot{\zeta}$ in this system:

$$\dot{\zeta}\frac{\partial z}{\partial \zeta}=-\int_{\zeta_H}^{\zeta}\left[\nabla_\zeta\cdot\left(\mathbf{V}\frac{\partial z}{\partial \zeta}\right)+\frac{\dfrac{\partial p_T}{\partial t}+J}{\gamma p}\frac{\partial z}{\partial \zeta}-\frac{Q}{c_pT}\frac{\partial z}{\partial \zeta}\right]d\zeta. \tag{7.14}$$

The top pressure tendency $\partial p_T/\partial t$ is determined so that (7.14) satisfies the upper boundary condition (7.7) and leads to

$$\frac{\partial p_T}{\partial t}=-\frac{\displaystyle\int_{\zeta_H}^{\zeta_T}\left[\frac{J}{\gamma p}\left(\frac{\partial z}{\partial \zeta}\right)-\frac{Q}{c_pT}\left(\frac{\partial z}{\partial \zeta}\right)+\nabla_\zeta\cdot\left(\mathbf{V}\frac{\partial z}{\partial \zeta}\right)\right]d\zeta}{\displaystyle\frac{1}{\gamma}\int_{\zeta_H}^{\zeta_T}\frac{1}{p}\left(\frac{\partial z}{\partial \zeta}\right)d\zeta}. \tag{7.15}$$

By combining the equation of state (2.5) and the hydrostatic equation (7.5), we determine temperature T from

$$T=-\frac{g}{R\left(\dfrac{\partial \zeta}{\partial z}\right)\dfrac{\partial}{\partial \zeta}\ln p}. \tag{7.16}$$

In this system, the prognostic variables are $\mathbf{V}$, p, and p_T which are predicted by (7.2), (7.8), and (7.15). The diagnostic variables $\dot{\zeta}$ and T are computed from (7.14) and (7.16).

We should comment here on the present formulation. If we assume (7.1) as ζ, we get $\partial\zeta/\partial z=-1/(z_T-H)$. However, it is important to point out that the present formulation is quite general for any variable ζ as long as: 1) the relationship between ζ and z is monotonic for fixed horizontal coordinates, 2) $\partial\zeta/\partial z$ is independent of time, and 3) $\partial\zeta/\partial z$ is continuous and "smooth" in the model.

The vertically integrated total energy equation in the ζ-system becomes identical to (5.15) in the z-system after rewriting the limits of integration by transforming the integration variable z to ζ.

8. Prediction equations in isentropic coordinates

Another vertical coordinate system especially convenient for description of adiabatic motions is the potential temperature or isentropic coordinate. For adiabatic motions, the potential temperature θ of an air parcel is conserved; thus, the pattern of isentropic surfaces reflects the vertical structure of the adiabatic atmosphere. The use of isentropic coordinates is not limited to adiabatic motions. Here we present prediction equations in the θ-system with the heating term included in the thermodynamic equation. As pointed out in Section 5, the formulation of prediction equations in isentropic coordinates is different from that in isobaric and height coordinates because of the requirement of an additional upper boundary condition to conserve total energy in the model.

Since $\nabla\theta$ vanishes on isentropic surfaces, logarithmic differentiation of (2.4a) yields

$$\nabla_\theta(\ln\theta)=\nabla_\theta\ln T-\frac{R}{c_p}\nabla_\theta\ln p=0. \tag{8.1}$$

Therefore, it follows that

$$\frac{1}{\rho}\nabla_\theta p=c_p\nabla_\theta T. \tag{8.2}$$

Thus, by defining

$$M=c_pT+gz \tag{8.3}$$

which is called the *Montgomery potential*, after Montgomery (1937) who first derived the expression, the horizontal equation of motion in the θ-system is reduced from (3.14) to

$$\frac{d\mathbf{V}}{dt}+f\mathbf{k}\times\mathbf{V}=-\nabla_\theta M+\mathbf{F} \tag{8.4}$$

where

$$\frac{d}{dt}=\left(\frac{\partial}{\partial t}\right)_\theta+\mathbf{V}\cdot\nabla_\theta+\dot{\theta}\frac{\partial}{\partial\theta}.$$

It is convenient to introduce the *Exner function*

$$\pi=c_p\left(\frac{p}{p_0}\right)^\kappa, \tag{8.5}$$

after Eliassen and Kleinschmidt (1957), where $\kappa=R/c_p$ and $p_0=1013$ mb. Using this function, we express the potential temperature (2.4a) by

$$\theta=c_pT/\pi, \tag{8.6}$$

the Montgomery potential (8.3) by

$$M=\theta\pi+gz, \tag{8.7}$$

and the thermodynamic equation (2.4) by

$$\dot{\theta}=d\theta/dt=Q/\pi. \tag{8.8}$$

Combining logarithmic differentiation of (2.4a) and the hydrostatic equation (2.2) and using (8.3) and (8.7), we obtain

$$\frac{\partial M}{\partial\theta}=\pi. \tag{8.9}$$

This is a convenient form of the hydrostatic equation in the θ-system. The Montgomery potential M is obtained by integrating (8.9) with respect to θ from the surface potential temperature θ_H at $z=H$:

$$M=M_H+\int_{\theta_H}^{\theta}\pi d\theta, \tag{8.10}$$

and M_H denotes the Montgomery potential at the earth's surface:

$$M_H=\theta_H\pi_H+gH \tag{8.11}$$

where π_H denotes the Exner function at the earth's surface. The continuity equation in the θ-system is obtained from (3.21) by replacing s with θ:

$$\frac{\partial}{\partial\theta}\left(\frac{\partial p}{\partial t}\right)_\theta+\nabla_\theta\cdot\left(\mathbf{V}\frac{\partial p}{\partial\theta}\right)+\frac{\partial}{\partial\theta}\left(\dot{\theta}\frac{\partial p}{\partial\theta}\right)=0. \tag{8.12}$$

The upper and lower boundary conditions in the θ-system corresponding to (4.1) and (4.3) are

$$\left.\begin{aligned}&\dot{\theta}=0 && \text{at } \theta=\theta_T=\text{const.}\\ &\dot{\theta}=\frac{\partial\theta_H}{\partial t}+\mathbf{V}_H\cdot\nabla\theta_H && \text{at } \theta=\theta_H\end{aligned}\right\} \tag{8.13}$$

where θ_T denotes the potential temperature of the model top. The upper boundary condition $\dot{\theta}=0$ requires the heating term to vanish. This is actually fairly well fulfilled around the 150-mb level in the atmosphere.

The value of potential temperature at the earth's surface θ_H is determined from the lower boundary condition in (8.13) by substituting (8.8) for $\dot{\theta}$:

$$\frac{\partial\theta_H}{\partial t}=-\mathbf{V}_H\cdot\nabla\theta_H+(Q/\pi)_H. \tag{8.14}$$

By integrating (8.12) with respect to θ from some level θ to the top θ_T and using the upper boundary condition in (8.13), we have

$$\left(\frac{\partial p}{\partial t}\right)_\theta=\frac{\partial p_T}{\partial t}+\int_\theta^{\theta_T}\nabla_\theta\cdot\left(\mathbf{V}\frac{\partial p}{\partial\theta}\right)d\theta-\dot{\theta}\left(\frac{\partial p}{\partial\theta}\right). \tag{8.15}$$

If we evaluate the above equation at $\theta=\theta_H$, exchange the integration and the differentiation noting that θ_H is a function of time and space, and use (5.10) applied to the θ-coordinate and the lower boundary condition in (8.13), we obtain the surface pressure tendency equation

$$\frac{\partial p_H}{\partial t}=\frac{\partial p_T}{\partial t}+\nabla\cdot\int_{\theta_H}^{\theta_T}\mathbf{V}\frac{\partial p}{\partial\theta}d\theta. \tag{8.16}$$

As discussed in Section 5, we need an additional upper boundary condition (5.19) or (5.20) in the θ-system. If we choose (5.19), we simply set $\partial p_T/\partial t=0$ in (8.15) and (8.16).

On the other hand, if we select (5.20), z_T is set as a constant and we must calculate $\partial p_T/\partial t$ at $\theta=\theta_T$. If (8.10) is evaluated at $\theta=\theta_T$, we have

$$\theta_T\pi_T+gz_T=\theta_H\pi_H+gH+\int_{\theta_H}^{\theta_T}\pi d\theta \tag{8.17}$$

where π_T denotes the value of π at $\theta=\theta_T$. Since z_T and H do not depend on time, by differentiating (8.17) with time, exchanging the integration and the differentiation noting that θ_H is a function of time, and using definition (8.5) for the Exner function, we obtain

$$\frac{\theta_T\pi_T}{p_T}\frac{\partial p_T}{\partial t}=\frac{\theta_H\pi_H}{p_H}\frac{\partial p_H}{\partial t}+\int_{\theta_H}^{\theta_T}\frac{\pi}{p}\left(\frac{\partial p}{\partial t}\right)_\theta d\theta. \tag{8.18}$$

Substitution of (8.15) and (8.16) into (8.18) and use of (8.8) for $\dot{\theta}$ yield the top pressure tendency

$$\frac{\partial p_T}{\partial t}=\frac{\left(\frac{\theta\pi}{p}\right)_H\nabla\cdot\int_{\theta_H}^{\theta_T}\mathbf{V}\frac{\partial p}{\partial\theta}d\theta+\int_{\theta_H}^{\theta_T}\left[\frac{\pi}{p}D(\theta)-\frac{Q}{p}\left(\frac{\partial p}{\partial\theta}\right)\right]d\theta}{\left(\frac{\theta\pi}{p}\right)_{z_T}-\left(\frac{\theta\pi}{p}\right)_H-\int_{\theta_H}^{\theta_T}\frac{\pi}{p}d\theta}, \tag{8.19}$$

where

$$D(\theta)=\int_{\theta}^{\theta_T} \nabla_\theta\cdot\left(\mathbf{V}\frac{\partial p}{\partial\theta}\right)d\theta. \tag{8.19a}$$

Equation (8.19) corresponds to (2.16b) in Kasahara and Washington (1967) for the z-system.

The initial conditions for this scheme require the fields of $\mathbf{V}$ and p on isentropic surfaces and also the surface potential temperature θ_H and surface pressure p_H. The fields of π and M on isentropic surfaces are obtained from (8.5) and (8.10). The field of $\dot\theta$ can be computed from (8.8) with one of the upper boundary conditions being $\dot\theta=0$. We can then predict the fields of $\mathbf{V}$ and θ_H from (8.4) and (8.14), and the fields of p and p_H from (8.15) and (8.16) with either $\partial p_T/\partial t=0$ or $\partial p_T/\partial t$ computed from (8.19).

Eliassen and Raustein (1968, 1970) performed numerical integrations of prediction equations in the isentropic coordinate system with an additional upper boundary condition, $\partial p_T/\partial t=0$. They dealt with adiabatic motions so that their prediction equations are identical to those in this section if $\dot\theta$ terms are set equal to zero. They also described a finite-difference procedure to deal with the lower boundary condition on a flat earth. Shapiro (1973) recently performed numerical integrations of prediction equations similar to those of Eliassen and Raustein using 20 isentropic layers to simulate frontogenesis. Eliassen and Rekustad (1971) applied isentropic prediction equations for the study of lee waves. In their model, the earth's orography was included, but the ground surface was assumed to coincide with an isentropic surface, thereby simplifying treatment of the earth's orography.

The isentropic coordinate system has computational disadvantages similar to those of the pressure coordinate system in the vicinity of the mountains. Since the surface potential temperature varies in space and time at the earth's surface, the lower boundary conditions are cumbersome to code. This difficulty may be avoided by introducing a new vertical coordinate η:

$$\eta=\frac{\theta_T-\theta}{\theta_T-\theta_H}, \tag{8.20}$$

where θ_T is a constant potential temperature at the top and θ_H denotes a variable potential temperature at the earth's surface. We assume that θ is a monotonic increasing function with height and hence $\theta_T>\theta_H$. We then have $\eta=1$ at $\theta=\theta_H$ and $\eta=0$ at $\theta=\theta_T$.

Logarithmic differentiation of potential temperature (2.4a) and the expression of potential temperature (8.6) with $p=\rho RT$ gives

$$\nabla_\eta\theta=\frac{1}{\pi}\left(c_p\nabla_\eta T-\frac{1}{\rho}\nabla_\eta p\right). \tag{8.21}$$

By combining the above with $\nabla_\eta\theta=\eta\nabla\theta_H$, which can be obtained by applying ∇_η to (8.20), it follows that

$$\frac{1}{\rho}\nabla_\eta p=-c_p\nabla_\eta T+\pi\eta\nabla\theta_H. \tag{8.22}$$

The horizontal equation of motion in this system is reduced from (3.14) to

$$\frac{d\mathbf{V}}{dt}+f\mathbf{k}\times\mathbf{V}=-\nabla_\eta M+\pi\eta\nabla\theta_H+\mathbf{F}, \tag{8.23}$$

where M denotes the Montgomery potential defined by (8.3),

$$\frac{d}{dt}=\left(\frac{\partial}{\partial t}\right)_\eta+\mathbf{V}\cdot\nabla_\eta+\dot\eta\frac{\partial}{\partial\eta}$$

and

$$\dot\eta=\frac{1}{\theta_T-\theta_H}(\eta\dot\theta_H-\dot\theta) \tag{8.24}$$

in which $\dot\theta_H$ and $\dot\theta$ are computed from (8.8).

The hydrostatic equation in this system is transformed from (8.9) to

$$\frac{\partial M}{\partial\eta}=-(\theta_T-\theta_H)\pi. \tag{8.25}$$

By integrating the above with respect to η from the earth's surface $\eta=1$, it follows that

$$M=M_H+(\theta_T-\theta_H)\int_\eta^1 \pi d\eta \tag{8.26}$$

where M_H denotes the surface value of M given by (8.11).

The continuity equation is obtained from (3.21) by replacing s with η:

$$\frac{\partial}{\partial\eta}\left(\frac{\partial p}{\partial t}\right)_\eta+\nabla_\eta\cdot\left(\mathbf{V}\frac{\partial p}{\partial\eta}\right)+\frac{\partial}{\partial\eta}\left(\dot\eta\frac{\partial p}{\partial\eta}\right)=0. \tag{8.27}$$

By integrating the above with respect to η from the top $\eta=0$ and using the upper boundary condition $\dot\eta=0$ at $\eta=0$, we obtain

$$\left(\frac{\partial p}{\partial t}\right)_\eta=\frac{\partial p_T}{\partial t}-\int_0^\eta \nabla_\eta\cdot\left(\mathbf{V}\frac{\partial p}{\partial\eta}\right)d\eta-\dot\eta\left(\frac{\partial p}{\partial\eta}\right). \tag{8.28}$$

Extending the integration down to the earth's surface $\eta=1$ and using the lower boundary condition $\dot\eta=0$, we get the surface pressure tendency equation

$$\frac{\partial p_H}{\partial t}=\frac{\partial p_T}{\partial t}-\nabla\cdot\int_0^1 \mathbf{V}\frac{\partial p}{\partial\eta}d\eta. \tag{8.29}$$

As an additional upper boundary condition, we select either $\partial p_T/\partial t$ as zero or z_T as constant so that

$\partial p_T/\partial t$ can be calculated by an equation similar to (8.19) in the θ-system.

9. Conclusions

We reviewed in this paper various prediction models based on the primitive equations using different vertical coordinate systems. In numerical models, we require the conservation of mass and total energy in the atmosphere of a finite depth. The formulation of prediction equations used a generalized vertical coordinate taking any well behaved, single-valued monotonic function of geometrical height.

The conservation of total energy in the height and pressure coordinate systems is achieved by the same upper and lower boundary conditions satisfying the conservation of mass. Other than these two systems, the conservation of total energy in a general vertical coordinate requires an additional upper boundary condition that either the top pressure is independent of time or the geopotential of the top coordinate surface is independent of time. The former condition is simpler to apply and the prediction equations become closer to those in the pressure coordinate. The latter condition requires the calculation of top pressure tendency similar to that in the height coordinate. It is inconsistent to assume both conditions simultaneously in general. Other than the pressure and height systems, the isentropic coordinate system is discussed in detail. Density is another possible variable suitable to a vertical coordinate and the formulation of the prediction equations follows very closely that of the isentropic coordinate.

The pressure, height, and potential temperature coordinate systems all have a common difficulty in handling the earth's orography. In p and θ, the variables vary not only along the earth's surface, but also with time. Thus, the location of the earth's surface relative to the coordinate surfaces is constantly changing. It is a substantial coding problem in computer logic to test for the location of the earth's surface in the finite-difference grid. Similar difficulty exists in identifying the earth's surface in the height coordinate system. However, the location of the earth's surface in the grid is invariant with time in the height coordinate system and coding of a special routine to handle the orography is simplified.

Considering the nature of advanced computers designed for processing arrays of data simultaneously, it is advantageous to apply transformed systems so that the earth's surface becomes a coordinate surface. We have already pointed out an inherent difficulty connected with the calculation of pressure gradient force in the sigma-system. We must anticipate similar difficulties in the transformed height and isentropic systems. Thus, the reduction of truncation errors in the evaluation of pressure gradient force on the transformed coordinate systems remains a problem.

Relative merits of various vertical coordinate systems are not discussed in this review, since the question involves practical matters such as the vertical resolution of the finite-difference scheme and the vertical structure of weather phenomena to be described with those systems.

The selection of the "best" vertical coordinate system, if it exists, is an important objective of future research in numerical weather prediction. For this purpose, it is desirable to write a computer code flexible enough that various prediction schemes with different vertical coordinate systems can be adopted with minimal changes in the program.

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APPENDIX

List of Symbols

A	any scalar function
c	variable representing x, y, or t for partial differentiation
c_p	specific heat at constant pressure
c_v	specific heat at constant volume
$D(\theta)$	defined by (8.19a)
$\mathbf{F}$	frictional force per unit mass
f	Coriolis parameter
g	earth's gravity
H	height of the earth's surface from mean sea level
$\mathbf{i}, \mathbf{j}, \mathbf{k}$	unit vectors in x-, y-, and z-coordinates
J	defined by (7.10)
k	horizontal kinetic energy per unit mass (5.1a)
M	$(=c_pT+gz)$ Montgomery function
p	pressure
P_*	$=p_H-p_T$
p_0	standard pressure, 1013 mb
Q	rate of heating per unit mass
R	$(=c_p-c_v)$ specific gas constant
s	generalized vertical coordinate
t	time
T	temperature
u, v	x- and y-components of $\mathbf{V}$
$\mathbf{V}$	horizontal velocity
w	vertical velocity
x, y, z	Cartesian coordinates directed eastward, northward and upward

Greek symbols

η	transformed isentropic coordinate
γ	$=c_p/c_v$
κ	$=R/c_p$
ω	$=dp/dt$

π Exner function (8.5)
ρ density
σ $=(p-p_T)/P_*$
θ potential temperature
ζ transformed height coordinate

Other symbols

∇ horizontal del operator
d/dt total derivative
$(\dot{\ })$ $=d(\ )/dt$

Subscripts (not defined above)

H evaluated at the earth's surface
T evaluated at the model top

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