Virtual Temperature

The ideal gas law (in molar form) is

$$P = n\, R^*\, T \tag{1}$$

where P is pressure, n is the molar number density of molecules in the gas in number of moles per unit volume and R^* is in Joules/Kelvin/mole (in mks units) and T is in kelvin. Notice pressure is proportional to number density (***not*** mass density). Still, in Earth (as opposed to planetary) atmospheric science, we tend to see this written as

$$P = nR^*T = nm\frac{R^*}{m}T = \rho RT \tag{2}$$

The mass density is

$$\rho = nm$$

Since the atmosphere is made up of different types of atoms and molecules, the m is the average mass of the molecules and atoms in the atmosphere. Therefore, when we introduce mass into the ideal gas law we must do some averaging over the different masses and recognize that the ratios of the masses are going to vary as the water vapor concentration varies.

$$\overline{m} = \frac{\sum_i n_i m_i}{\sum_i n_i} \tag{3}$$

$$P = \sum_i n_i m_i \frac{R^*}{m_i} T_i = T \sum_i \rho_i R_i \tag{4}$$

The temperature comes outside of the sum because we are assuming the gas is in thermodynamic equilibrium. So all constituents have the same temperature.

The issue is the amount of water vapor in the atmosphere is variable and has a molecular mass of 18 g/mole which is less massive than the mass per mole of the dry gas of 28.97 grams per mole. Below the homopause (~90 km), the dry part of the atmosphere is well mixed so the mixing ratios of the major dry constituents are fixed. We can therefore divide the atmosphere into a dry part and a wet part.

$$P = T\sum_i \rho_i R_i = \left(\rho_{dry}R_{dry} + \rho_{wet}R_{wet}\right)T = \left(\rho_{dry}R + \rho_{wet}R_v\right)T \tag{5}$$

Now, density shows up in the hydrostatic relation where hydrostatic pressure is proportional to mass because the gravitational force is proportional to mass

$$dP = -g\, \rho\, dz$$

The virtual temperature concept, T_v, comes from wanting simplify the ideal gas law by pretending that all of the molecules are dry molecules and adjust the temperature to get the pressure to come out correctly using R ($=R_{\text{dry}}$) while at the same time keeping the total density correct to satisfy the hydrostatic equation.

$$P = \left(\rho_{dry}R_d + \rho_{wet}R_{wet}\right)T = \rho R_d T_v \tag{6}$$

where ρ is the total mass density.

So, to determine the equation for the virtual temperature in terms of the total pressure and the partial pressure of water vapor, *e*, we start by writing the total density in terms of the partial pressures

$$\rho = \left(\rho_{dry} + \rho_{wet}\right) = \frac{(P-e)}{R_d T} + \frac{e}{R_v T} \tag{7}$$

Then we plug (7) into (6) and solve for the virtual temperature

$$\frac{P}{R_d T_v} = \frac{(P-e)}{R_d T} + \frac{e}{R_v T}$$

$$PT = \left[\frac{(P-e)}{R_d} + \frac{e}{R_v}\right] R_d T_v$$

$$T = \left[\frac{(P-e)}{P} + \frac{eR_d}{PR_v}\right] T_v = \left[\left(1 - \frac{e}{P}\right) + \frac{em_v}{Pm_d}\right] T_v = \left[\left(1 - \frac{e}{P}\right) + \frac{e}{P}\varepsilon\right] T_v$$

$$T_v = \frac{T}{\left[1 + \frac{e}{P}(\varepsilon - 1)\right]} \tag{8}$$

where $\varepsilon = m_v/m_d = 0.622$.