Up to this point, we have been primarily concerned with atmospheric *optics*. There, the main goal has been to calculate k_e, ω_0, and $p(\cos\Theta)$ of a parcel of air as a function of the properties of the gases and particles within.

The General Radiative Transfer Equation for Plane Parallel Atmospheres

Now, we are prepared to tackle the more complex problem of radiative transfer, in which we consider an atmosphere with known optics, k_e, ω_0, and $p(\cos\Theta)$ as functions of position in space ($\mathbf{x}$), and determine the light field within it, defined by the intensity, $I(\mathbf{x},\mathbf{\Omega})$, which is also a function of all possible directions of propagation, $\mathbf{\Omega}$. We will need boundary conditions of intensity, usually specified at the TOA and surface. We assume the radiation transfer process is instantaneous – that is, any change in the radiative field with time is due to the change in the boundary conditions, and not due to the time it takes to set up an equilibrium field within. We also assume: 1) all radiation can be treated as unpolarized, 2) the phase function depends on scattering angle only, which implicitly means that all scatterers are randomly oriented or spherical. Fair enough.

Let me remind you of the general radiative transfer equation:

$$\frac{dI(\mathbf{\Omega})}{ds} = -k_e I(\mathbf{\Omega}) + k_e \frac{\omega_0}{4\pi}\int_0^{2\pi} d\varphi' \int_{-1}^{1} I(\mathbf{\Omega}')\, p(\mathbf{\Omega}\cdot\mathbf{\Omega}')\, d\mu' + k_e \frac{\omega_0}{\pi} B_\lambda(T)$$

where *ds* is a infinitesimal path in the direction of light propagation, $\mathbf{\Omega}$. The first term on the RHS is the extinction of the beam due to absorption and scattering to other directions. The second term may look intimidating, but it simply accounts for light scattered from other directions into the direction of interest. The third term is the emission by the layer. For the present discussion, we will restrict ourselves to the solar part of the atmospheric spectrum, and ignore atmospheric emission term. Note that $\mathbf{\Omega}\cdot\mathbf{\Omega}'$ is just the cosine of the scattering angle, $\cos(\Theta)$.

If we make a third assumption, that the atmospheric properties, k_e, ω_0, $p(\Theta)$ are only functions of altitude, and thus do not vary horizontally, things simplify a bit. In this case, we define a new vertical coordinate system, $\tau(z)$, based on the extinction coefficient profile in the atmosphere, $k_e(z)$.

$$d\tau = -k_e dz$$

$$ds = \frac{dz}{\mathbf{\Omega}\cdot\hat{\mathbf{z}}}$$

Note that $\mathbf{\Omega}\cdot\mathbf{z}$ is simply $\cos(\theta)$, the cosine of the polar angle of propagation.

It becomes easier to think about solar radiative transfer if, when we switch from $z \rightarrow \tau$, we also switch our coordinate system from thinking about direction of *propagation* to direction of *incidence*. We define $\mu = -\cos(\theta)$, where μ is the cosine of the polar angle of *incidence*, and θ is the polar angle of *propagation*. (This way μ is positive for radiation propagating in the $+\tau$ direction.)

$$k_e ds = -\frac{d\tau}{\mathbf{\Omega}\cdot\hat{\mathbf{z}}}$$

$$\frac{1}{k_e}\frac{dI}{ds} = \mu\frac{dI}{d\tau}$$

Thus for plane parallel atmospheres we have

$$\frac{dI(\mathbf{\Omega})}{d\tau} = -\frac{1}{\mu}I(\mathbf{\Omega}) + \frac{1}{\mu}\frac{\omega_0}{4\pi}\int_0^{2\pi} d\varphi' \int_{-1}^{1} I(\mathbf{\Omega}')\,p(\mathbf{\Omega}\cdot\mathbf{\Omega}')\,d\mu' + \frac{1}{\mu}\frac{\omega_0}{\pi}B_\lambda(T) \qquad (1)$$

This is a complicated *integro-differential* equation. It has no known general solution. However, we can make some additional approximations and simplify it.

The Two Stream Approximation

For many plane parallel applications, the angular distribution of radiation doesn't change radically from layer-to-layer in the atmosphere. For example, within a thick cloud, the light field is nearly isotropic, meaning that variations in I with $\mathbf{\Omega}$ are very slight. The two-stream approximation takes advantage of this: it treats the full light field as consisting of only two streams – an upward stream, and a downward stream. Commonly, we think of the downward stream as the downward irradiance, F_D, (W m^{-2}), and the upward stream as the upward irradiance, F_U, (W m^{-2}). Irradiance is simply the cosine-weighted integral over a hemisphere,

$$F_D = \int_0^{2\pi} d\varphi \int_0^{1} I(\mathbf{\Omega})\mu d\mu$$

$$F_U = \int_0^{2\pi} d\varphi \int_0^{-1} I(\mathbf{\Omega})\mu d\mu$$

So what we do in (one version of) the two stream approximation, is treat the downward and upward irradiances as though they were collimated radiances propagating at some effective angles with incidence angle cosines, μ_0 and $-\mu_0$, respectively. In this case, scattered radiation has only two directions to go: up or down. This simplifies the integral in (1) considerably. If we consider that some fraction, *f*, of the scattering from a beam goes forward (back into the same stream), and the remaining (1-*f*) of the scattering goes backwards (into the other stream), we end up with just two equations – one for the upward stream, and one for the downward stream, instead of an independent equation for each direction $\mathbf{\Omega}$, like we had in (1). The two-stream equations are as follows:

$$\frac{dF_D}{d\tau} = -\frac{1}{\mu_0} F_D + \frac{1}{\mu_0}\omega_0 f F_D + \frac{1}{\mu_0}\omega_0 (1-f) F_U \tag{2a}$$

$$\frac{dF_U}{d\tau} = -\frac{1}{-\mu_0} F_U + \frac{1}{-\mu_0}\omega_0 f F_U + \frac{1}{-\mu_0}\omega_0 (1-f) F_D \tag{2b}$$

These two equations for two unknowns can be solved straightforwardly. We first define two quantities that we'll need for the solutions:

$$F_N = \int_0^{2\pi} d\varphi \int_{-1}^{1} I(\mathbf{\Omega}) \mu d\mu = F_D - F_U$$

$$F_A = \int_0^{2\pi} d\varphi \int_{-1}^{1} I(\mathbf{\Omega}) d\mu = \left(F_D + F_U\right) / \mu_0$$

The net flux, F_N, is the net energy transport across a surface, accounting for the upward and downward streams. The actinic flux, F_A, is a measure of the total light intensity averaged over all directions. It is the quantity used when calculating *j* rates in photochemistry. Think of F_N as the net radiation crossing a plane of unit area, and F_A as the radiation incident on a sphere of unit area. Using this notation, (2a,b) can be linearly recombined to become:

$$\frac{dF_N}{d\tau} = -F_A + \omega_0 f F_A + \omega_0 (1-f) F_A \tag{3a}$$

$$\mu_0^2 \frac{dF_A}{d\tau} = -F_N + \omega_0 f F_N - \omega_0 (1-f) F_N \tag{3b}$$

which can be reduced to

$$\frac{dF_N}{d\tau} = -F_A (1-\omega_0) \tag{4a}$$

$$\mu_0^2 \frac{dF_A}{d\tau} = -F_N (1-\omega_0 g) \tag{4b}$$

We have introduced the approximation that $f = (1/2 + g/2)$, where g is the asymmetry parameter (average cosine of scattering; 1st moment of the phase function; see Mie Scattering notes).

Equations (4a,b) represent the solution to the 2-stream approximation, based on the assumptions we've put together so far. Most often, we take $\mu_0 = 0.5$ to best represent the situation for isotropic scattering. (Consider the flux-weighted average pathlength through a thin layer given isotropic incident radiation – you can derive it yourself.)

Interpretation of 2-stream equations

(4a) can be interpreted as the *net radiative flux divergence*. It is the difference between the net flux entering a layer and the net flux exiting the layer from the other side. If the layer doesn't absorb, ($\omega_0 = 1$) we see that F_N is a constant. Since absorption of radiation removes energy from the radiation field, F_N decreases through layers where $\omega_0 < 1$. The rate of decrease is equal to the amount of radiation available to the layer for absorption, which is simply F_A – absorption doesn't care whether the radiation is propagating upward or downward.

The interpretation of (4b) is a bit less intuitive. It tells us that actinic flux decreases in the direction of the net flux (unless $\omega_0 g = 1$: which is the case if there is no absorption and all scattering is in the forward direction – as if there were no extinction at all!). Net flux represents a flow of radiation from a source towards an absorber. So (4b) means the actinic flux tends to decrease as you get closer to that absorber. Imagine a cloud layer over the ocean. It's brightest above the cloud, where you get the radiation from the sun above and the bright reflection from the cloud below. If you then drop through the cloud to the dark ocean surface, it's much darker there – even though the net flux – the downward minus the upward flux – may not have changed much.

Solution of 2-stream equation for no-absorption case ($\omega_0 = 1$)

The case for clouds in the visible part of the spectrum is well approximated by this case. (5a,b) reduce to…

$$\frac{dF_N}{d\tau} = 0 \tag{6a}$$

$$\mu_0^2 \frac{dF_A}{d\tau} = -F_N(1-g) \tag{6b}$$

and integrating yields two algebraic equations with two unknowns

$$F_N = F_{N0} \tag{7a}$$

$$F_A = F_{A0} - F_{N0}\frac{(1-g)}{\mu_0^2}\tau \tag{7b}$$

This simply reiterates that the net flux is constant where there's no absorption, and the actinic flux decreases with optical depth, depending on the value of the net flux.

We need two boundary conditions to solve for our two unknowns. Let's consider the case where the sun is shining on a cloud layer of optical thickness τ_0 over a surface with reflectivity, R_S.

$$F_{D0} = \mu_0 S_0 \tag{8a}$$

$$F_U(\tau_0) = R_S F_D(\tau_0) \tag{8b}$$

These boundary conditions are in terms of the upward and downward irradiances, not in terms of the net and actinic fluxes that the solutions provide. To use these boundary conditions, we need to invert F_N and F_A to get back to F_D and F_U.

$$F_D = \frac{1}{2}\left(\mu_0 F_A + F_N\right) \tag{9a}$$

$$F_U = \frac{1}{2}\left(\mu_0 F_A - F_N\right) \tag{9b}$$

Using (7a,b), we end up with

$$F_D = \frac{1}{2}\left(\mu_0 F_{A0} - F_{N0}(1-g)\frac{\tau}{\mu_0} + F_{N0}\right) \tag{10a}$$

$$F_U = \frac{1}{2}\left(\mu_0 F_{A0} - F_{N0}(1-g)\frac{\tau}{\mu_0} - F_{N0}\right) \tag{10b}$$

Using (8a) to eliminate F_{A0}, we end up with

$$F_D = \mu_0 S_0 - F_{N0}(1-g)\frac{1}{2}\frac{\tau}{\mu_0} \tag{11a}$$

$$F_U = \mu_0 S_0 - F_{N0} - F_{N0}(1-g)\frac{1}{2}\frac{\tau}{\mu_0} \tag{11b}$$

Using (8b) to find F_{N0}, we end up with

$$F_{N0} = \frac{\mu_0 S_0 \left(1 - R_S\right)}{\left[1 + \left(1 - R_S\right)(1-g)\frac{1}{2}\frac{\tau_0}{\mu_0}\right]} \tag{12}$$

Let's pause deriving for a moment to look at this solution for the net flux (which is constant throughout the column). We see that in the limit that the surface reflectivity = 1, there is no net flux – how could there be if there are no absorbers? What goes in comes out. In the case that the surface reflectivity is 0 (perfect absorption), then we have the net flux depending on the total cloud optical depth. For this case, in the limit that there is no cloud, then we have net flux equaling the solar input – everything incident is absorbed. As optical depth gets larger (and the cloud reflects more to space), the net flux reduces until the limit that the cloud is very, very thick in which case net flux goes to zero, meaning the cloud acts as a perfectly reflective surface, irrespective of what is beneath it.

Using (12) with (11a,b), we have expressions for upward and downward radiation, from which we can get the column reflectivity, F_{U0}/F_{D0}. (If these were broadband calculations, we would call the column reflectivity the albedo).

$$F_D = \mu_0 S_0 \frac{1 + (1 - R_S)(1 - g)\frac{1}{2}\frac{\tau_0 - \tau}{\mu_0}}{1 + (1 - R_S)(1 - g)\frac{1}{2}\frac{\tau_0}{\mu_0}} \tag{13a}$$

$$F_U = \mu_0 S_0 \frac{R_S + (1 - R_S)(1 - g)\frac{1}{2}\frac{\tau_0 - \tau}{\mu_0}}{1 + (1 - R_S)(1 - g)\frac{1}{2}\frac{\tau_0}{\mu_0}} \tag{13b}$$

Some intuition checks on 13a. Is the downward flux equal to the solar input at the TOA (where τ=0?). Check. Does it decrease as you get closer to the surface? Check – but only if $R_S < 1$. If $R_S = 1$, then F_D is a constant. That makes sense, actually. See if you can justify to yourself why… Intution checks on 13b. Does it equal the surface reflectivity as $\tau_0 \rightarrow 0$? Check. Does it equal F_D as $R_S \rightarrow 1$? Check.

Now we can calculate the reflectivity of the column at the TOA

$$R_C = \frac{R_S + (1 - R_S)(1 - g)\frac{1}{2}\frac{\tau_0}{\mu_0}}{1 + (1 - R_S)(1 - g)\frac{1}{2}\frac{\tau_0}{\mu_0}} \tag{14}$$

This is the standard solution for column albedo in the limit of no absorption. You often see it in a different form, with the assumption that the surface reflectivity is 0.

$$R_C = \frac{\tau_0}{\tau_0 + \frac{2\mu_0}{(1-g)}} \tag{15a}$$

An analogous form (equivalent to 14) that does not assume surface reflectivity is zero is given by

$$R_C = \frac{R_S \frac{2\mu_0}{(1-g)(1-R_S)} + \tau_0}{\tau_0 + \frac{2\mu_0}{(1-g)(1-R_S)}} \tag{15b}$$

Note that these are non-linear functions of cloud optical depth. They resemble neither the single-scattering limit (where reflectivity of a cloud layer is proportional to optical depth times the upscatter fraction) nor the Beer's law limit (where transmission decays exponentially with depth into the cloud).

Problem: A) Consider a cloud with optical depth $\tau_0 = 20$, $g = 0.85$, over a dark surface, $R_S = 0$. The direct solar beam (the intensity of the disc of the sun) decreases due to Beer's Law. However, because a large fraction of this decrease is due to scattering, the total downward flux F_D does not decrease at the same rate. Use the single-scattering solution above to estimate the percent reduction in downward flux in the middle of the cloud ($\tau = 10$) relative to that at the top of the atmosphere. Compare this to the percent reduction in the intensity of the direct solar beam at $\tau = 10$ compared to that at the top of the atmosphere.

Solution: (13a) is used to solve for F_D at $\tau = 10$ and $\tau = 0$. One minus their ratio (times 100%) yields the percent reduction in downward flux. The quantity $(1-g)/2\mu_0$ is 0.15. We have % reduction = 1- (1 + .15(10))/(1 + .15(20)) = 1 - 2.5/3.5 = 28%. In contrast, the direct solar beam is attenuated by Beer's law. the percent reduction is $1 - \exp(-\tau/\mu_0)$ = 100%. Only two-billionths of the direct solar radiation penetrates this deep into the cloud.

This is an example of the effect of multiple scattering. The brightness of a cloud doesn't depend so much on that first scattering event between the solar photons and the cloud. Rather, the photons enter the cloud and start scattering around like pinballs in a pachinko machine. Using this photon analogy, the *light transport through a multiple-scattering medium becomes more like an equilibrium diffusion problem* than a direct transport problem. All the downward photons at $\tau = 10$ have scattered numerous times before reaching that depth. And not all of these photons cascaded directly downward to this level – they all follow a random walk. Many of these photons have crossed back and forth across this level multiple times.

It is often asserted that the reason the radiation penetrates so deeply into the cloud is due to the forward scattering effect – i.e. $g > 0$. This isn't true.

Problem: B) Consider the case where all of our cloud drops are perfect mirrors that reflect all extinguished radiation back where it came from. This would be the case where $g = -1$. Calculate the percent reduction in downward flux at $\tau = 10$.

Solution: In this case, $(1-g)/2\mu_0$ is 2 instead of 0.15. We have % reduction = 1- (1 + 2(10))/(1 + 2(20)) = 1 - 21/41 = 49%. There is still significant penetration of radiation deep into the cloud.

Even though each droplet sent a photon back where it came from – in the reverse direction – we STILL get a significant penetration of radiation down to deep optical depths where only 2 in a billion photons get there without scattering. So it's clearly not the fact that the scattering is in the forward direction. It's because scattering is a random walk problem, and if you continually throw photons at the top of the cloud (as the sun does), you will only reflect a fraction of them initially – the rest will begin randomly walking their way through the medium. For example, suppose a photon enters the cloud and scatters at the average optical depth of 0.5. It heads back up, but has a significant chance of scattering again before getting back out the top. If it does so it will scatter back downward again – the g = -1 case is a double edged sword. Sure, it increases the probability that a downward photon will be scattered back up on its first scattering event. But it also decreases the probability that an upward photon will make it out without being sent back downward again. In fact, if you look at Eqs. (13-15), we see that the term $2\mu_0/(1-g)(1-R_S)$ acts as a scaling optical depth against which the multiple scattering properties are measured. Small (and negative) values of g simply shorten this depth, making the scattering more effective per unit optical depth. Surface albedo plays a role in this scaling depth too. Why? Remember the 2-stream solution is an equilibrium one. As we've seen it's a diffusion problem. Equilibrium problems are set by their boundary conditions. The surface reflectivity is a boundary condition in this problem. Note that in the limit that the surface is perfectly reflective, the scaling optical depth goes to infinity. That simply says that at each level in the cloud, the upflux and the downflux are constant – the properties of the entire system are not dependent on the thickness of the cloud. As an example, put a perfectly reflective cloud above a perfectly reflective surface, and you don't see a difference from above or within. (This is why it's hard to tell where clouds are over the arctic ice caps – this has been a persistent problem for satellite remote sensing estimates of the effects of clouds on surface radiation budget).

We can now ask – how many times does the average photon scatter before reaching this optical depth of 10? The average path it takes between scattering events is about μ_0 units of optical depth (0.5 in our case). For g = 0.85, f = 0.925, meaning that only 7.5% of the scattering events are backward. So lets ignore 85% of the optical depth as being pure forward scattering, and treat the other 15% as being a random walk event – equal probability of forward and backward scattering. In this case, we scale our mean path length from 0.5 to 0.5/0.15 = 3.3 units of optical depth, comprising 6.7 forward scattering events per random scattering event. To reach an optical depth of 10, there must be about 3 more "random" forward scattering events than backwards ones. This will typically happen after 3^2 or 9 random events. Since for each random event there are 6.7 pure forward scattering events, the total # of scatterings is 9*6.7 = 60. So a photon typically scatters 60 times before reaching an optical depth of 10.

Solution of 2-stream equation for absorptive cases (ω_0 < 1)

The reason that multiple-scattering solutions allow penetration of photons to high optical depths is because there's no loss of energy in each scattering event. Suppose the singlescattering albedo was 0.99, meaning that each scattering event resulted in a loss of 1% of the energy (or for the purists, a 1% probability of the photon being absorbed instead of scattered). In this case, our back of the envelope estimate of 60 scattering events to reach an optical depth of 10 leads to a reduction in photon intensity to exp(-.6) = 54%. That is, only half of the photons destined for an optical depth of 10 will actually make it before being absorbed on the way down – even for this very low absorption per scattering ratio of 0.01.

The point here is that even very slight single-scattering-albedos can produce large absorptions if the medium is a multiple scattering one. This result actually comes right out of the 2-stream equations if we allow $\omega_0 < 1$). To solve this, we have to go back to Eqs. (4a,b)

$$\frac{dF_N}{d\tau} = -F_A(1-\omega_0) \tag{4a}$$

$$\mu_0^2\frac{dF_A}{d\tau} = -F_N(1-\omega_0 g) \tag{4b}$$

Taking a 2[nd] derivative of (4a) equation and substituting in (4a,b) yields

$$\begin{aligned}
\frac{d^2F_N}{d\tau^2} &= F_N\frac{(1-\omega_0)(1-\omega_0 g)}{\mu_0^2} \\
F_N &= F_{N-}\exp\left(-\sqrt{(1-\omega_0)(1-\omega_0 g)}\frac{\tau}{\mu_0}\right) + F_{N+}\exp\left(\sqrt{(1-\omega_0)(1-\omega_0 g)}\frac{\tau}{\mu_0}\right) \\
F_A &= \frac{1}{\mu_0}\sqrt{\frac{(1-\omega_0 g)}{(1-\omega_0)}}F_{N-}\exp\left(-\sqrt{(1-\omega_0)(1-\omega_0 g)}\frac{\tau}{\mu_0}\right) \\
&\quad -\frac{1}{\mu_0}\sqrt{\frac{(1-\omega_0 g)}{(1-\omega_0)}}F_{N+}\exp\left(\sqrt{(1-\omega_0)(1-\omega_0 g)}\frac{\tau}{\mu_0}\right)
\end{aligned}$$

This is a very different-looking solution than the conservative one. It looks a bit more like Beer's Law due to the exponential terms.

We still have two unknowns and need two boundary conditions. The solution gets very ugly when we sub in the surface boundary conditions, so we will replace the surface boundary condition with the so-called "*semi-infinite atmosphere*" solution, meaning that the cloud is infinitely optically thick – it has no lower boundary. (You can only do this if there is absorption – otherwise the photons will keep diffusing downward ad infinatum). To get reasonable conditions for large optical depth, we must have $F_{N+} = 0$. In this case, (16) reduces to

$$\begin{aligned}
F_N &= F_{N-}\exp\left(-\sqrt{(1-\omega_0)(1-\omega_0 g)}\frac{\tau}{\mu_0}\right) \\
F_A &= \frac{1}{\mu_0}\sqrt{\frac{(1-\omega_0 g)}{(1-\omega_0)}}F_{N-}\exp\left(-\sqrt{(1-\omega_0)(1-\omega_0 g)}\frac{\tau}{\mu_0}\right)
\end{aligned}$$

Note that this semi-infinite atmosphere solution has the net flux decreasing exponentially with optical depth, but not as quickly as Beer's law reduces intensity unless $\omega_0 = 0$, in which case the transport equations (2a,b) are entirely reduced to Beer's law. You can see that in the limit that $\omega_0 \rightarrow 1$, the net flux becomes constant – as we had in the conservative solution. Note that you'd have to use L'Hopital's rule to evaluate what F_A

does in the limit that $\omega_0 \rightarrow 1$. A hint is that (for this semi-infinite atmosphere) $F_{N-} \rightarrow 0$. This is also consistent with the conservative solution for very large optical depth.

We'll define the quantity $\sqrt{(1-\omega_0)/(1-\omega_0 g)}$ as the similarity parameter, *s*. This make the notation hereout simpler.

To solve for F_{N-} using upper boundary condition, (and otherwise to relate the solution to our upward and downward fluxes) we need to use (9a,b)

$$F_D = \frac{1}{2}\left(\mu_0 F_A + F_N\right) \tag{9a}$$

$$F_U = \frac{1}{2}\left(\mu_0 F_A - F_N\right) \tag{9b}$$

which yield

$$F_D = \frac{1}{2} F_{N-}\left[\frac{1}{s}+1\right]\exp\left(-\sqrt{(1-\omega_0)(1-\omega_0 g)}\,\frac{\tau}{\mu_0}\right)$$

$$F_U = \frac{1}{2} F_{N-}\left[\frac{1}{s}-1\right]\exp\left(-\sqrt{(1-\omega_0)(1-\omega_0 g)}\,\frac{\tau}{\mu_0}\right)$$

Using the same upper boundary condition as before, we have

$$F_{N-} = 2\mu_0 S_0\left[\frac{1}{s}+1\right]^{-1}$$

which yields

$$F_D = \mu_0 S_0 \exp\left(-\sqrt{(1-\omega_0)(1-\omega_0 g)}\,\frac{\tau}{\mu_0}\right)$$

$$F_U = \mu_0 S_0 \frac{[1-s]}{[1+s]}\exp\left(-\sqrt{(1-\omega_0)(1-\omega_0 g)}\,\frac{\tau}{\mu_0}\right)$$

These equations obey our intuition – that the downward flux scales like Beer's law, but scales deeper due to multiple scattering effects, and the upward flux is simply a fixed fraction of the downward flux at all levels. (It's a semi-inifinite atmosphere, so no matter where you are in the cloud, the properties of the column below you look the same.) The reflectivity of the column is

$$R_C = \frac{[1-s]}{[1+s]}$$

This is an important parameter, since we can readily see R_C from space. For thick atmospheres, such as those of the gas giants, we see that the reflectivity at a given wavelength depends just on the sensitivity parameter. Consider the case where absorption is very weak. For this case, we usually refer to the co-albedo $\varpi = 1 - \omega_0$. For $\varpi << 1$, $s \rightarrow \sqrt{\varpi/(1-g)}$. Doing a linearization yields

$$R_C = 1 - 2\sqrt{\varpi/(1-g)}.$$

This means that the absorptivity of the column is

$$\alpha_C = 2\sqrt{\varpi/(1-g)}$$

Note that in the single scattering limit, absorptivity of a layer goes like $\varpi d\tau$. But in the multiple scattering limit, absorptivity scales like the square root of ϖ. This means that absorption ramps up extremely quickly as ϖ exceeds zero. This is purely a consequence of the fact that – in a thick cloud – there are so many scattering events that even the smallest absorption will have a huge effect. As absorption goes up, however, the number of scattering events goes down, and so the effect tapers off.

Consider the varied colors of Jupiter. It doesn't actually take much absorption by a thick cloud to produce such colors. Suppose the giant red spot has an albedo (reflectivity) in the Red part of the spectrum of 0.99 and an albedo in the blue part of the spectrum of 0.7, giving it its characteristic red hue (I'm making these numbers up out of thin air, mind you). We'll also assume that the clouds are made of 10 μm particles like Earth's clouds, so $(1-g) = 0.15$. From our similarity solution, we infer that the single-scattering albedo in the red is 0.999996, and that in the blue is .997. So even when 99.7% of extinction events are scattering rather than absorption, the reflectivity of a thick cloud still drops to as low as 0.7!. This is the power of multiple scattering on absorption.

Appendix: The Two Stream Solution using the Eddington approximation.

OK. Let's start again with the plane-parallel radiative transfer equation.

$$\frac{dI(\mathbf{\Omega})}{d\tau} = -\frac{1}{\mu} I(\mathbf{\Omega}) + \frac{1}{\mu}\frac{\omega_0}{4\pi}\int_0^{2\pi} d\varphi' \int_{-1}^{1} I(\mathbf{\Omega}')\, p(\mathbf{\Omega}\cdot\mathbf{\Omega}')\, d\mu' + \frac{1}{\mu}\frac{\omega_0}{\pi} B_\lambda(T) \qquad (1)$$

Another way to approach the two stream approximation is to simulate the irradiance field with the following parameterization:

$$I(\mu) = I_0 + I_1\mu \qquad \text{(A.1)}$$

I_0 represents an isotropic component to the radiance field, and I_1 represents a directional component that is downward (towards $+\tau$) when positive. If you think about it, it's not a bad approximation, since you can expand any function in spherical coordinates with the proper Legendre Polynomials. This is just the first two terms of such a polynomial. Note that this result yields

$$F_N = \int_0^{2\pi} d\varphi \int_{-1}^{1} I(\mathbf{\Omega})\mu d\mu = \frac{4}{3}\pi I_1$$

$$F_A = \int_0^{2\pi} d\varphi \int_{-1}^{1} I(\mathbf{\Omega}) d\mu = 4\pi I_0$$

$$F_D = \int_0^{2\pi} d\varphi \int_0^{1} I(\mathbf{\Omega})\mu d\mu = \pi I_0 + \frac{2}{3}\pi I_1 = \frac{1}{2}\left(\mu_0 F_A + F_N\right)$$

$$F_U = \int_0^{2\pi} d\varphi \int_0^{-1} I(\mathbf{\Omega})\mu d\mu = \pi I_0 - \frac{2}{3}\pi I_1 = \frac{1}{2}\left(\mu_0 F_A - F_N\right)$$

Note that μ_0 didn't come from these equations. I just put it in there (substituting for a factor of ½) to show that the relationships between (F_D and F_U) and (F_N and F_A) are identical to those we obtained previously when we assumed $\mu_0 = 0.5$.

The only thing left to do is put our expression for I into the radiative transfer equation, integrate over all directions, and see what falls out. To get equations for I_0 and I_1, we first do an unweighted integration, and then do an integration weighting by μ. To do this properly, we must first find an expression for p(Θ) that is integrable. For consistency with the intensity parameterization, we also specify:

$$\begin{aligned} p(\mu) &= p_0 + p_1\mu \\ &= 1 + 3g\mu \end{aligned} \quad \text{(A.2)}$$

If you crank through the integrals *very carefully*, you'll find you're left with the fairly simple expressions

$$\frac{dI_0}{d\tau} = -I_1(1-\omega_0 g) \quad \text{(A.3a)}$$

$$\frac{dI_1}{d\tau} = -3I_0(1-\omega_0) \quad \text{(A.3b)}$$

Using the expressions derived above to relate I_0 and I_1 to the actinic and net fluxes, respectively, we end up with:

$$\frac{dF_A}{d\tau} = -3F_N(1-\omega_0 g) \quad \text{(A.4a)}$$

$$\frac{dF_N}{d\tau} = -F_A(1-\omega_0) \quad \text{(A.4b)}$$

Note that the Eddington approximation is nearly identical to (4a,b). The only difference is that we've replaced μ_0^2 (= 1/4) with a factor of 1/3. This corresponds to an effective "stream angle" of 55° (instead of the 60° used before).

The rest of this Eddington solution follows identically to the two-stream case, except we replace $\mu_0 = 0.5$ for the two-stream case with $\mu_0 = 3^{-1/2} = 0.58$ in the Eddington case.

Other common methods to solve the radiative transfer equation

Other useful methods exist. The discrete ordinates method is similar to the 2-stream method, but is implemented with a large number of radiation streams, each representing a finite range of solid angle. A technique called "Gaussian quadrature" is used to choose and integrate the streams.

A particularly powerful approach is that of the Monte Carlo Model. Like its name suggests, it is a statistical solution that utilizes a large number of randomizations to achieve a representative result. Essentially, a "packet" of photons is traced as is flows through the atmosphere. Beer's law is interpreted as a probabilistic function. First, a random number between 0 and 1 is chosen. The absolute value of the natural log of this number (which will range from 0 to infinity) can be equated to the optical depth that this packet will travel before interacting with the atmosphere. When the packet interacts, a fraction $(1 - \omega_0)$ is considered to be absorbed. The remaining photons in the packet are scattered. This is also handled probabilistically. The phase function is integrated and inverted. Another random number from 0 – 1 is used with this inverted phase function to find the angle that the packet will scatter towards. Then the process begins again as the packet moves in the new, scattered direction. This procedure is repeated until the packet is virtually completely absorbed or leaves the top of the atmosphere. Then the whole process is repeated for millions of more photon packets. Statistics are accumulated of photons crossing specific levels and being absorbed, so that layer fluxes and heating rates can be tallied at the end.