Vorticity: The Mesoscale Perspective

The mesoscale requires us to think about vorticity in 3 dimensions, and as we'll see, how it is related to buoyancy.

Our starting point is the 3-D vorticity equation (curl of wind)

$$\vec{\omega} = \nabla \times \vec{V} \qquad \vec{V} = (u, v, w)$$

Horizonatal and vertical components:

$$\xi = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \qquad \hat{i} \text{ direction}$$

$$\eta = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \qquad \hat{j} \text{ direction}$$

$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \qquad \hat{k} \text{ direction.}$$

As we said before, ξ and η are now relavant for mesoscale, whereas before (WAF I) we only cared about ζ for synoptic scales.

What we eventually want is a vorticity tendency equation in the form of $\frac{d\vec{\omega}}{dt}$.

Will provide physical insight into sources of vorticity generation on both synoptic and mesoscales.

Starting point is full 3-D conservation of momentum equation:

$$\frac{d\vec{V}}{dt} = \underbrace{-\frac{1}{\rho}\nabla p}_{\text{PGF}} \underbrace{-2\vec{\Omega}\times\vec{V}}_{\text{Coriolis}} \underbrace{+g\hat{k}}_{\text{gravity}} \underbrace{+\vec{F}}_{\text{Friction}} \quad (1)$$

Where

$$\frac{d\vec{V}}{dt} = \underbrace{\frac{\partial\vec{V}}{\partial t}}_{\text{Local time rate of change}} + \underbrace{\vec{V}\cdot\nabla\vec{V}}_{\text{Advection term}} \quad (2)$$

* From this point on, deriving the final equation for $\partial\vec{\omega}/\partial t$ involves vector manipulation in a derivation. Suggest review vector identities if getting a little lost in math...

Use these vector identities for advective and gravity terms.

Advection $\quad \vec{V}\cdot\nabla\vec{V} = \nabla\frac{|\vec{V}|^2}{2} + \vec{\omega}\times\vec{V} \qquad (3)$

gravity $\quad g = -\nabla(gz) \qquad (4)$

Substitute back into momentum eqn.

$$\frac{\partial\vec{V}}{\partial t} + \underbrace{\nabla\left(\frac{|\vec{V}|^2}{2} - gz\right)}_{\text{Advection + gravity}} + \vec{\omega}\times\vec{V}$$

$$= \underbrace{-\frac{1}{\rho}\nabla p}_{\text{PGF}} \; \underbrace{- f\hat{k}\times\vec{V}}_{\substack{\text{Coriolis}\\ \text{(simplified)}}} \; \underbrace{+ \vec{F}}_{\text{Friction.}} \qquad (5)$$

For simplicity, we'll proceed on from here neglecting friction, though it is included in the book.

Also assume that the 'w' related components in the Coriolis term can be neglected.

Take $\nabla \times$ (Eq. 5) to get local time rate of change equation for $\vec{\omega}$

$$\frac{\partial \vec{\omega}}{\partial t} + \nabla \times \vec{\omega} \times \vec{V} = -\nabla\left(\frac{1}{\rho}\right) \times \nabla p + \nabla \times \left(-f\hat{k} \times \vec{V}\right) \tag{6}$$

Note, at this step, we got rid of some terms by vector identities:

$$\nabla \times \left(\nabla\left(\frac{|\vec{V}|^2}{2} - gz\right)\right) = 0 \tag{7}$$

$$\frac{1}{\rho} \nabla \times \nabla p = 0 \tag{8}$$

How to simplify?

① We can also use vector identities to rewrite 2nd term on LHS of (6)

$$\nabla \times \vec{\omega} \times \vec{V} = \vec{\omega}(\nabla \cdot \vec{V}) - \cancel{\vec{V}(\nabla \cdot \vec{\omega})} + (\vec{V} \cdot \nabla)\vec{\omega} - (\vec{\omega} \cdot \nabla)\vec{V} \tag{9}$$

The crossed out term goes to zero in (9) because

$$\vec{V}(\nabla \cdot \vec{\omega}) = \vec{V}(\nabla \cdot \nabla \times \vec{V}) = 0 \tag{10}$$

(2) Similarly for Coriolis term in (6)

$$\nabla \times (-f\hat{k} \times \vec{V}) = -f\hat{k}(\nabla \cdot \vec{V}) - (\vec{V} \cdot \nabla) f\hat{k}$$

$$+ \underbrace{\cancel{\vec{V}(\nabla \cdot f\hat{k})}}_{\text{to zero}} + f(\hat{k} \cdot \nabla)\vec{V} \qquad (11)$$

(3) For density term

$$-\nabla \rho^{-1} = \rho^{-2} \nabla \rho \qquad (12)$$

Modified resultant equation (6) for $\partial \vec{\omega}/\partial t$

$$\frac{\partial \vec{\omega}}{\partial t} + \underbrace{\cancel{(\vec{\omega} + f\hat{k})(\nabla \cdot \vec{V})}}_{\text{to zero}} + (\vec{V} \cdot \nabla)(\vec{\omega} + f\hat{k})$$

$$= \left[(\vec{\omega} + f\hat{k}) \cdot \nabla \right] \vec{V} + \frac{1}{\rho^2} \nabla \rho \times \nabla p \qquad (13)$$

(4) Assume that atmosphere is incompressible

$$\nabla \cdot \vec{V} = 0 \qquad (14)$$

So 2^{nd} term on LHS → to zero.

Now finally get interpretable equation for $d\vec{\omega}/dt$ (from circled steps 1-4)

$$\underbrace{\frac{\partial \vec{\omega}}{\partial t}}_{\text{Local time rate of change}} + \underbrace{\vec{V} \cdot \nabla (\vec{\omega} + f\hat{k})}_{\text{Advection of vorticity}}$$

$$= \underbrace{\left[(\vec{\omega} + f\hat{k}) \cdot \nabla \right] \vec{V}}_{\text{Stretching / tilting term}}$$

$$+ \frac{1}{\rho^2} \nabla\rho \times \nabla p \qquad (15)$$

Note ρ and p vary much more in vertical than horizontal!

Baroclinic vorticity generation - due to buoyancy! (B)

Relevant only for $\vec{\omega}_h$!

More on baroclinic generation term in (15)

$$\frac{1}{\rho^2} \nabla\rho \times \nabla p \approx \nabla \times B\hat{k} \qquad (16)$$

Buoyancy

$$\nabla \times B\hat{k} = \frac{\partial B}{\partial y}\hat{\imath} - \frac{\partial B}{\partial x}\hat{\jmath} \qquad (17)$$

→ Get contributions to horizontal components of $\vec{\omega}$ via spatial horizontal variations in B.

Vorticity tendency equation decomposition:

Horizontal

$\hat{x}$

$$\frac{\partial \xi}{\partial t} = -\vec{V}\cdot\nabla\xi + \vec{\omega}\cdot\nabla u + f\frac{\partial u}{\partial z} + \frac{\partial B}{\partial y} \quad (18)$$

Adv. | Vortex Stretching/tilting term | Tilting of planetary vort | Baroclinic generation

$\hat{y}$

$$\frac{\partial \eta}{\partial t} = -\vec{V}\cdot\nabla\eta + \vec{\omega}\cdot\nabla v + f\frac{\partial v}{\partial z} - \frac{\partial B}{\partial x} \quad (19)$$

Adv. | Vortex Stretching/tilting term | Tilting of planetary vort | Baroclinic generation

Vertical

$\hat{z}$

$$\frac{\partial \zeta}{\partial t} = -\vec{V}\cdot\nabla(\zeta+f) + \vec{\omega}\cdot\nabla w + f\frac{\partial w}{\partial z} \quad (20)$$

Adv. | Vortex Stretching, tilting term | Tilting of planetary vort.

<u>Note</u>: As little comparative variation in ρ and p in horizontal vs. vertical, buoyancy effects contribute <u>only</u> to horizontal vorticity generation!

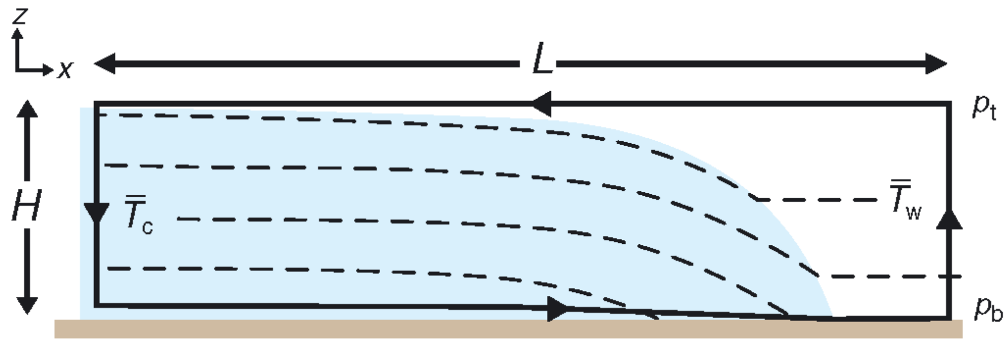

Figure 2.4 An example of how Bjerknes' circulation theorem can be applied to a mesoscale air mass boundary associated with a horizontal temperature gradient. The relatively cold air mass is shaded blue, and dashed lines indicate surfaces of constant density. The closed solid line is the loop about which the circulation is to be evaluated (its quasihorizontal segments follow constant pressure surfaces). The mean virtual temperature in the warm air mass along the circuit is $\overline{T}_w$, and in the cold air mass it is $\overline{T}_c$.

$$\frac{d\eta}{dt} = -\frac{\partial B}{\partial x} = -\frac{g}{\overline{T}_v}\frac{\partial T_v'}{\partial x}, \tag{2.106}$$

The vorticity vector (eta) pointing into page

Finally, if we have a flow-normal coordinate system, can re-orient the horizontal vorticity tendency equations

Crosswise vorticity → Flow perpindicular

StreamWise vorticity → Flow parallel

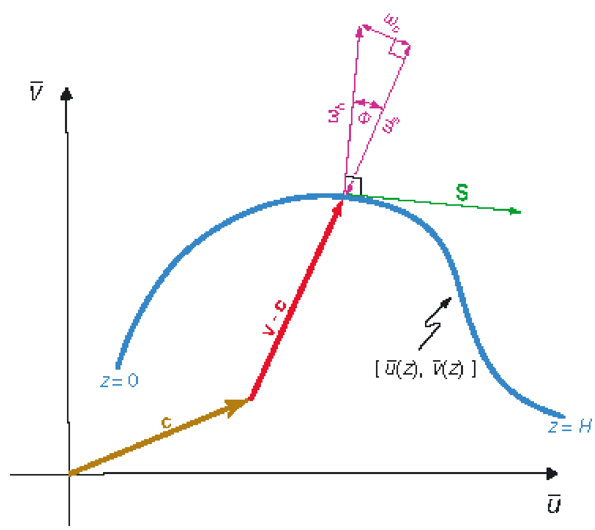

Figure 2.12 Diagram of a hodograph $[\overline{u}(z), \overline{v}(z)]$ depicting the storm motion vector $\mathbf{c}$, storm-relative wind vector $\mathbf{v} - \mathbf{c}$, shear vector $\mathbf{S}$, and environmental horizontal vorticity vector $\boldsymbol{\omega}_{\mathrm{h}}$. When $\mathbf{S}$ is a good representation of the thermal wind (i.e., when winds are close to geostrophic), $\boldsymbol{\omega}_{\mathrm{h}}$ points toward the cold air and $\mathbf{v} \cdot \boldsymbol{\omega}_{\mathrm{h}}$ is porportional to temperature advection. The streamwise and crosswise vorticity components, ω_{s} and ω_{c}, respectively, are also indicated.

Which terms in $\partial\vec{\omega}/\partial t$ matter for synoptic & mesoscales?

Advection of ζ → More on Synoptic scales, related to QG-forced vertical motion.

Advection of ξ, η → More on mesoscales

Vortex stretching → Both synoptic and mesoscales
- diabatic heating
- terrain changes
- storm updrafts

Vortex tilting → More mesoscale
- tilt horizontal vorticity into storm updraft

Baroclinic term → More mesoscale
- density variations in horizontal due to storm inflows and outflows.

Viscosity (friction) → More mesoscale
- locally increase vorticity in rotating updraft??

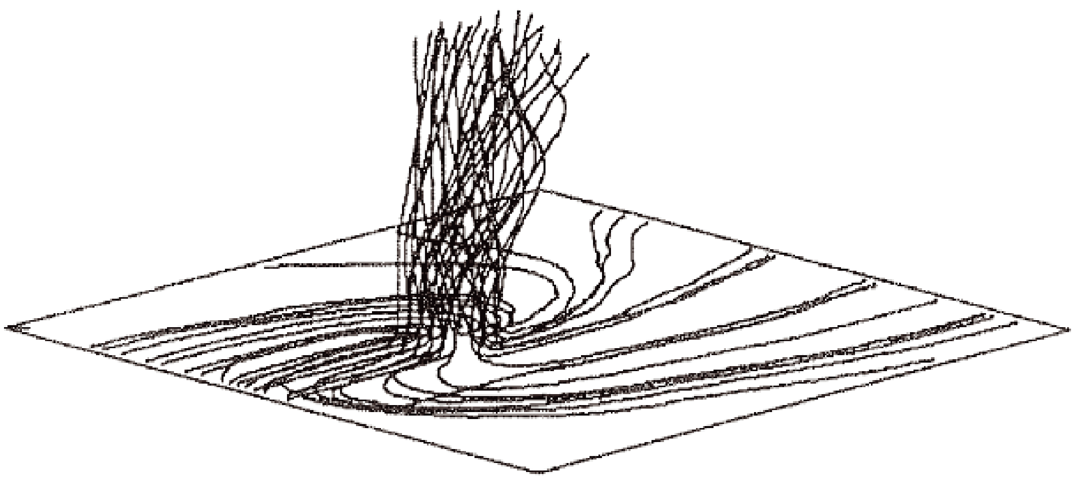

Figure 2.5 Vortex lines associated with a tornado in a simulation by Walko (1993). Courtesy of the American Geophysical Union.