Dynamically consistent, quasi-hydrostatic equations for global models with a complete representation of the Coriolis force

By A. A. WHITE and R. A. BROMLEY*

Meteorological Office, UK

(Received 1 July 1993; revised 24 May 1994)

SUMMARY

The spherical polar components of the Coriolis force consist of terms in \( \sin \phi \) and terms in \( \cos \phi \), where \( \phi \) is latitude (referred to the frame-rotation vector as polar axis). The \( \cos \phi \) Coriolis terms are not retained in the usual hydrostatic primitive equations of numerical weather prediction and climate simulation, their neglect being consistent with the shallow-atmosphere approximation and the simultaneous exclusion of various small metric terms. Scale analysis for diabatically driven, synoptic-scale motion in the tropics, and for planetary-scale motion, suggests that the \( \cos \phi \) Coriolis terms may attain magnitudes of order 10% of those of key terms in the hydrostatic primitive equations. It is argued that the \( \cos \phi \) Coriolis terms should be included in global simulation models.

A global, quasi-hydrostatic model having a complete representation of the Coriolis force is proposed. Conservation of axial angular momentum and potential vorticity, as well as energy, is achieved by a formulation in which all metric terms are retained and the shallow-atmosphere approximation is relaxed. Distance from the centre of the earth is replaced by a pseudo-radius which is a function of pressure only. This model is put forward as a more accurate alternative to the traditional hydrostatic primitive equations; it preserves the desired conservation laws and may be integrated by broadly similar grid-point methods.

1. INTRODUCTION

Numerical models for weather prediction and global climate seek to simulate the behaviour of the atmosphere by using accurate representations of the governing equations of motion, thermodynamics and continuity. These equations contain terms describing intrinsic fluid dynamical effects such as advection and the pressure-gradient force, and terms representing sources or sinks of momentum and heat. The latter may be referred to as forcing terms; they represent the divergence of subgrid-scale fluxes, and diabatic forcing due to radiative flux divergence and latent heating.

Over the past 30 years much attention has been paid to improving the representation of the forcing processes, especially in the context of climate simulation using global circulation models. Over the same period much progress has also been made in refining time-integration schemes and in accommodating increased spatial resolutions. The emergence of competitively economic spectral models has been a notable development. There has, however, been no change in the assumed representation of the intrinsic fluid dynamical processes: almost all global models are based on the hydrostatic primitive equations (HPEs). (The exceptions are models based on various geostrophically balanced approximations of the HPEs. These models are used for the important scientific purpose of developing comprehension of atmospheric behaviour, and the more sophisticated of them may offer forecast accuracy rivalling that of HPE models.)

The HPEs are simpler in several respects than the complete equations of motion. In addition to the neglect of vertical accelerations in the momentum balance, the HPEs use a spherical approximation to the spheroidal geometry of geopotential surfaces, and assume the shallow-atmosphere approximation. Various metric terms are neglected. Of greater quantitative importance, the Coriolis terms involving \( 2\Omega \cos \phi \), which appear in the zonal and vertical components of the momentum equation, are omitted. (Here \( \Omega \) is the rotation rate of the earth and \( \phi \) is latitude.) The omission of these \( \cos \phi \) Coriolis terms is the ‘traditional approximation’, which has been a matter of controversy in the

* Corresponding author: Meteorological Office, London Road, Bracknell, Berkshire RG12 2SZ, UK.
background of dynamical meteorology for many years (see, for example, Eckart 1960; Phillips 1966, 1968, 1973; Veronis 1968 and Wangsness 1970). (A close relation exists between the shallow-atmosphere approximation, the neglect of certain metric terms, and the omission of the $\cos \phi$ Coriolis terms. Taken together, but not individually, they constitute a dynamically consistent approximation which implies satisfactory analogue forms of energy, angular momentum and potential vorticity conservation. The term ‘traditional approximation’ is customarily reserved for the omission of the $\cos \phi$ Coriolis terms, however.)

The $\cos \phi$ Coriolis terms have recently been considered in a number of studies in meteorology, oceanography and geophysical fluid dynamics. Leibovich and Lele (1985) included them in a comprehensive investigation of Ekman-layer stability, and Mason and Thomson (1987) retained them in a numerical simulation of boundary-layer eddies. Garwood et al. (1985) considered the importance of the $\cos \phi$ Coriolis terms in the turbulent kinetic-energy budget of the oceanic surface mixed layer. The model of planetary geostrophic motion proposed by Shutts (1989) includes the terms after consistent approximation of the Lagrangian function. Burger and Riphagen (1990) retained the terms (and others not included in the HPEs) in a study of the equations of motion expressed in an arbitrary vertical-coordinate system. Draghi (1987, 1989) argued that the $\cos \phi$ Coriolis terms represent the most important nonhydrostatic effect in mesoscale atmospheric systems. He proposed ways of including the terms in formulations which use tangent-plane and other geometric approximations to the earth’s sphericity; analogues of energy and potential-vorticity conservation laws are retained.

In this paper we examine the importance of the $\cos \phi$ Coriolis terms and conclude that they may be non-negligible in planetary-scale motion, and in tropical synoptic-scale motion in which diabatic processes are of first-order importance in the thermodynamic equation and are balanced by vertical advection of potential temperature. We recommend the inclusion of the $\cos \phi$ Coriolis terms in simulation models of the global atmosphere, but emphasize that the good conservation properties of the HPEs set a high standard against which any new set of equations should be judged. We then propose a dynamically consistent acoustically filtered model of a global, compressible atmosphere with the $\cos \phi$ Coriolis terms included. The proposed model involves relaxation of the hydrostatic approximation in the sense that certain terms other than those representing gravity and the vertical pressure gradient are retained in the vertical component of the momentum equation. This is accomplished (when pressure is used as vertical coordinate) through an adaptation of the procedure used by Miller (1974) to formulate a nonhydrostatic convection model in pressure coordinates.

Components of the Navier–Stokes equation and the corresponding HPE forms are reviewed in section 2, with special attention to conservation properties. Section 3 considers the quantitative and qualitative importance of the $\cos \phi$ Coriolis terms. The new equations are presented in section 4, and their conservation properties are established. A concluding discussion follows in section 5. A list of symbols and their definitions is given in the appendix.

This paper is based on two internal reports from the Meteorological Office—White et al. (1994), and White and Bromley (1994). The reader is referred to these reports for further details and comment.

2. THE NAVIER–STOKES EQUATION AND THE HYDROSTATIC PRIMITIVE EQUATIONS

The components of the Navier–Stokes equation in a spherical polar coordinate system are derived in standard texts (see, for example, Gill (1982)). The HPEs and their
conservation properties are discussed in detail by Lorenz (1967), Phillips (1973) and Hoskins et al. (1985). Aspects which are important in this study are reviewed here.

(a) The Navier–Stokes equation

If velocities \( u \) are measured relative to a system rotating with angular velocity \( \Omega \), the Navier–Stokes equation is

\[
\frac{Du}{Dt} + 2\Omega \times u - g + \frac{1}{\rho} \nabla p = F.
\]  

(2.1)

Here \( g \) is apparent gravity (true gravity plus the centrifugal term \(-\Omega \times (\Omega \times r)\), where \( r \) is position relative to the centre of the earth): \( g = -\nabla \Phi \), where \( \Phi \) is the geopotential, \( \rho \) is density, \( p \) is pressure and \( F \) is the frictional force per unit mass. The spheroidal geopotential surfaces are customarily represented as spheres. In spherical polar coordinates the three components of (2.1) are then

\[
\frac{Du}{Dt} = \left( 2\Omega + \frac{u}{r \cos \phi} \right) (v \sin \phi - w \cos \phi) + \frac{1}{\rho r \cos \phi} \frac{\partial p}{\partial \lambda} = F_{\lambda},
\]

(2.2)

\[
\frac{Dv}{Dt} = \left( 2\Omega + \frac{u}{r \cos \phi} \right) v \sin \phi + \frac{vw}{r} + \frac{1}{\rho r} \frac{\partial p}{\partial \phi} = F_{\phi},
\]

(2.3)

\[
\frac{ Dw}{Dt} = \left( 2\Omega + \frac{u}{r \cos \phi} \right) w \cos \phi + \frac{v^2}{r} + g + \frac{1}{\rho} \frac{\partial p}{\partial r} = F_{r},
\]

(2.4)

Here

\[
\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \frac{u}{r \cos \phi} \frac{\partial}{\partial \lambda} + \frac{v}{r} \frac{\partial}{\partial \phi} + \frac{w}{r} \frac{\partial}{\partial r} = \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla
\]

(2.5)

and \( u, v, w \) are the components of \( \mathbf{u} \) in the \( \lambda \) (longitude), \( \phi \) (latitude) and \( r \) (radial) directions; the polar axis is in the direction of \( \Omega \). The coordinate configuration is shown in Fig. 1.

When taken together with the continuity and thermodynamic equations

\[
\frac{D\rho}{Dt} + \rho \text{div} \mathbf{u} = 0
\]

(2.6)

\[
\frac{D\theta}{Dt} = \left( \frac{\theta}{T_{cp}} \right) Q
\]

(2.7)

Eqs. (2.2)–(2.5) imply the following conservation laws for axial angular momentum, energy and potential vorticity:

\[
\rho \frac{D}{Dt} \left( (u + \Omega r \cos \phi) r \cos \phi \right) = \rho F_{r} \cos \phi - \frac{\partial p}{\partial \lambda},
\]

(2.8)

\[
\rho \frac{D}{Dt} \left( \frac{1}{2} \mathbf{u}^2 + \Phi + c_{e} T \right) + \text{div}(p \mathbf{u}) = \rho (Q + \mathbf{u} \cdot \mathbf{F}),
\]

(2.9)

\[
\rho \frac{D}{Dt} \left( \frac{Z \cdot \nabla \theta}{\rho} \right) = Z \cdot \nabla \frac{D\theta}{Dt} + \nabla \theta \cdot \nabla \mathbf{F}.
\]

(2.10)

Perfect gas behaviour, \( p = \rho RT \), has been assumed. In Eq. (2.10), \( Z \) is the absolute vorticity, \( 2\Omega + \text{curl} \mathbf{u} \). The components of \( Z \) in the \((\lambda, \phi, r)\) system are
Figure 1. The \((\lambda, \phi, r)\) spherical polar system.

\[
\begin{align*}
Z_\lambda &= \frac{1}{r} \frac{\partial w}{\partial \phi} - \frac{1}{r} \frac{\partial}{\partial r} (ru) \\
Z_\phi &= 2\Omega \cos \phi + \frac{1}{r} \frac{\partial}{\partial r} (ru) - \frac{1}{r \cos \phi} \frac{\partial w}{\partial \lambda} \\
Z_r &= 2\Omega \sin \phi + \frac{1}{r \cos \phi} \left( \frac{\partial v}{\partial \lambda} - \frac{\partial}{\partial \phi} (u \cos \phi) \right)
\end{align*}
\]

The divergence of a vector field \(A = (A_\lambda, A_\phi, A_r)\) is

\[
\text{div} \ A = \frac{1}{r \cos \phi} \left( \frac{\partial A_\lambda}{\partial \lambda} + \frac{\partial}{\partial \phi} (A_\phi \cos \phi) \right) + \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 A_r)
\]

and the gradient of \(\theta\) is the vector

\[
\text{grad} \ \theta = \left( \frac{1}{r \cos \phi} \frac{\partial \theta}{\partial \lambda}, \frac{1}{r \cos \phi} \frac{\partial \theta}{\partial \phi}, \frac{\partial \theta}{\partial r} \right)
\]

The notation 'grad', 'div' and 'curl' will be reserved for the quantities defined by (2.13), (2.12) and (2.11). Shallow-atmosphere versions of the respective operators will be indicated by '\(\nabla\)'.

(b) **Height coordinate forms of the hydrostatic primitive equations**

The HPEs which correspond to Eqs. (2.2)-(2.4) are

\[
\frac{D u}{D t} - \left( 2\Omega + \frac{u}{a \cos \phi} \right) v \sin \phi + \frac{1}{\rho a \cos \phi} \frac{\partial \rho}{\partial \lambda} = F_\lambda
\]
The terms omitted are those in $2\Omega \cos \phi$ (the $\cos \phi$ Coriolis terms), four metric terms, the vertical acceleration $Dw/Dt$ and the vertical component $F$, of the frictional force per unit mass. The shallow-atmosphere approximation has also been made: $r$ has been replaced by $a$, the earth's mean radius, except in the derivative terms $\partial/\partial r$ which are retained as $\partial/\partial z$, $z$ being height above mean sea level. In Eq. (2.16) $g$ is to be understood as an appropriate mean magnitude of $g$, independent of position and time. The Coriolis terms remaining in (2.14) and (2.15) may be written in the familiar form $f\mathbf{k} \times \mathbf{v}$ if a horizontal vector equation is constructed. Here $f = 2\Omega \sin \phi$, $\mathbf{k}$ is the unit vector in the local vertical and $\mathbf{v}$ is the horizontal part of $\mathbf{u}$. Thus, although the Coriolis force acts in planes normal to the rotation axis, in the HPEs it is represented as a force lying in the local horizontal plane.

The HPE continuity and thermodynamic equations are

$$\frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{u} = 0$$

(2.18)

$$\frac{D\theta}{Dt} = \left( \frac{\theta}{Tc_p} \right) Q.$$  

(2.19)

Here $D/Dt$ is defined as in (2.17), and

$$\nabla \cdot \mathbf{u} = \frac{1}{a \cos \phi} \left( \frac{\partial u}{\partial \lambda} + \frac{\partial}{\partial \phi} (v \cos \phi) \right) + \frac{\partial w}{\partial z}.$$  

(2.20)

Analogues of the conservation properties (2.8)–(2.10) are implied by the HPEs. The axial angular-momentum principle is

$$\rho \frac{D}{Dt} \left( (u + \Omega a \cos \phi) a \cos \phi \right) = \rho F_a \cos \phi - \frac{\partial p}{\partial \lambda}$$

(2.21)

the energy-conservation law is

$$\rho \frac{D}{Dt} \left( \frac{1}{2} v^2 + g z + c_a T \right) + \nabla \cdot (pu) = \rho (Q + v \cdot F_h)$$

(2.22)

and the potential-vorticity law is

$$\rho \frac{D}{Dt} \left( \frac{\xi \cdot \nabla \theta}{\rho} \right) = \xi \cdot \nabla \left( \frac{D\theta}{Dt} \right) + \nabla \theta \cdot \nabla \times \mathbf{F}_h.$$  

(2.23)

Here

$$\nabla \theta = \left( \frac{1}{a \cos \phi}, \frac{1}{a \partial \phi}, \frac{\partial}{\partial \Sigma} \right)$$  

(2.24)
and

\[ \zeta = 2\Omega k \sin \phi + \nabla \times \mathbf{v} \]  
(2.25)

with

\[ \nabla \times \mathbf{v} = \left\{ -\frac{\partial v}{\partial z}, \frac{\partial u}{\partial z}, \frac{1}{a \cos \phi} \left( \frac{\partial v}{\partial \lambda} - \frac{\partial}{\partial \phi} (u \cos \phi) \right) \right\}. \]  
(2.26)

The conservation properties (2.21), (2.22) and (2.23) depend on the simultaneous application of the shallow-atmosphere approximation, the neglect of four metric terms and the neglect of the \( \cos \phi \) Coriolis terms. The omission of the \( \cos \phi \) Coriolis terms from the HPEs is motivated largely by a desire to preserve good conservation properties when the shallow-atmosphere approximation is made.

(c) **Pressure-coordinate forms of the hydrostatic primitive equations**

The principal quasi-hydrostatic model proposed in section 4 uses pressure (rather than height) as the vertical coordinate. For comparison we lay out in this section the pressure-coordinate forms of the HPEs and note the corresponding versions of the various conservation properties. The pressure-coordinate HPEs are precise transforms of the height-coordinate HPEs (Eqs. (2.14)-(2.20)).

The zonal and meridional components of the HPE momentum balance may be written as

\[ \frac{Du}{Dt} - \left( 2\Omega + \frac{u}{a \cos \phi} \right) v \sin \phi + \frac{1}{a \cos \phi} \frac{\partial \Phi}{\partial \lambda} = F_\lambda \]  
(2.27)

\[ \frac{Dv}{Dt} + \left( 2\Omega + \frac{u}{a \cos \phi} \right) u \sin \phi + \frac{1}{a \cos \phi} \frac{\partial \Phi}{\partial \phi} = F_\phi \]  
(2.28)

with

\[ \frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \frac{u}{a \cos \phi} \frac{\partial}{\partial \lambda} + \frac{v}{a \cos \phi} \frac{\partial}{\partial \phi} + \omega \frac{\partial}{\partial \phi}. \]  
(2.29)

Here \( \omega = \frac{Dp}{Dt}, \Phi = gz \) and the derivatives \( \partial/\partial t, \partial/\partial \lambda, \partial/\partial \phi \) are each taken at constant pressure.

The continuity equation is

\[ \nabla_p \cdot \mathbf{v} + \frac{\partial \omega}{\partial p} = 0 \]  
(2.30)

in which

\[ \nabla_p \cdot \mathbf{v} = \frac{1}{a \cos \phi} \left( \frac{\partial u}{\partial \lambda} + \frac{\partial}{\partial \phi} (v \cos \phi) \right) \]  
(2.31)

and the thermodynamic equation may be written as

\[ c_p \frac{DT}{Dt} - \frac{RT \omega}{p} = Q \]  
(2.32)

(with \( D/Dt \) given by (2.29)).
For many purposes it is helpful to work in terms of the pressure-based but height-like coordinate \( z_s(p) \) defined by

\[
z_s(p) = \int_p^{p_1} \frac{RT_s(p') \, dp'}{g \rho_s}
\]

(2.33)

in which \( T_s = T_s(p) \) is some reference temperature profile and \( p_1 \) is a constant reference surface pressure. In terms of \( z_s \), Eqs. (2.29) and (2.30) become

\[
\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \frac{u}{a \cos \phi} \frac{\partial}{\partial \lambda} + \frac{v}{a \partial \phi} + \frac{w}{a \partial z_s} \nabla_p \cdot \mathbf{v} + \frac{1}{\rho_s \partial z_s} (\rho_s \mathbf{w}) = 0
\]

(2.34)

(2.35)

in which

\[
\mathbf{w} = \frac{Dz_s}{Dt} = - \frac{\omega RT_s}{g \rho_p}
\]

(2.36)

and

\[
\rho_s = \frac{p}{RT_s},
\]

(2.37)

Equation (2.35) is isomorphic to the continuity equation of an incompressible fluid having a mean density profile \( \rho_s(z) \), where \( z \) is geometric height.

The HPE conservation laws (2.21)-(2.23) take the following forms in pressure coordinates. The axial angular-momentum principle is

\[
\frac{D}{Dt} \{(u + Q \cos \phi)a \cos \phi\} = F_a \cos \phi - \frac{\partial \Phi}{\partial \lambda}
\]

(2.38)

the energy-conservation law is

\[
\frac{D}{Dt} \left( \frac{1}{2} v^2 + c_p T \right) + \nabla_p \cdot (\mathbf{v} \Phi) + \frac{\partial}{\partial \rho} (\omega \Phi) = Q + \mathbf{v} \cdot \mathbf{F}_h
\]

(2.39)

and the potential-vorticity law is

\[
\rho_s \frac{D}{Dt} \left( \frac{\zeta \cdot \nabla \theta}{\rho_s} \right) = \zeta \cdot \nabla \left( \frac{D \theta}{Dt} \right) + \nabla \theta \cdot \nabla \times \mathbf{F}_h.
\]

(2.40)

Here

\[
\nabla = \left( \frac{\partial}{\partial \lambda}, \frac{1}{a \cos \phi} \frac{\partial}{\partial \phi}, \frac{1}{\partial z_s} \right)
\]

(2.41)

and

\[
\zeta = 2 \Omega k \sin \phi + \nabla \times \mathbf{v}
\]

(2.42)

with

\[
\nabla \times \mathbf{v} = \left\{ - \frac{\partial v}{\partial z_s}, \frac{\partial u}{\partial z_s}, \frac{1}{a \cos \phi} \left( \frac{\partial v}{\partial \lambda} - \frac{\partial}{\partial \phi} (u \cos \phi) \right) \right\}.
\]

(2.43)
Since \( p \) is here held constant in the horizontal derivatives, the operators \( \nabla \) and \( \nabla \times \) — and hence \( \xi \)—are differently defined than in the height-coordinate case (Eqs. (2.24)–(2.26)). We do not consider it desirable to introduce a special notation to emphasize this.

3. The Importance of the \( \cos \phi \) Coriolis Terms

Inaccuracies are inevitably introduced by the omissions (and geometric distortions) made in replacing the Navier–Stokes equations by the HPEs. Under a wide range of conditions the \( \cos \phi \) Coriolis terms are by far the largest of the omitted terms.

(a) Scale analysis of the zonal momentum balance

Consider the term \( 2\Omega w \cos \phi \) in Eq. (2.2) in relation to the material derivative \( Du/Dt \). In quasi-hydrostatic motion, continuity sets an upper bound on vertical velocities as

\[
W \leq UH/L
\]

(3.1)

where, respectively, \( W \) and \( U \) are vertical and horizontal velocity-scales, and \( H \) and \( L \) are vertical and horizontal length-scales. Taking \( Du/Dt \sim U^2/L \) gives

\[
|2\Omega w \cos \phi|/|Du/Dt| \approx 2\Omega H \cos \phi/U
\]

which is independent of \( L \). The condition for the neglect of \( 2\Omega w \cos \phi \) in comparison with \( Du/Dt \) in Eq. (2.2) is thus

\[
2\Omega H \cos \phi/U \ll 1
\]

(3.2)

or

\[
2\Omega U \cos \phi/g \ll U^2/gH.
\]

(3.3)

With \( \Omega = 2\pi \) radians per day, \( H = 10^4 \) m and \( U = 10 \) m s\(^{-1}\), the quantity \( 2\Omega H \cos \phi/U \) takes a value of about 0.14 \( \cos \phi \).

Standard scale analysis shows that free, synoptic-scale motion is quasi-nondivergent in the sense that \( W \leq UH/L \) (Phillips 1963; Pedlosky 1987). In such motion \( W \) is typically one order of magnitude less than \( UH/L \): hence the upper bound (3.1) is not approached, and it is safe to neglect the term \( 2\Omega w \cos \phi \) in Eq. (2.2).

For other scales of motion, and in different circumstances, the situation is less clear-cut. From a scale analysis of the momentum and thermodynamic equations, Charney (1963) concluded that free synoptic-scale motion in the tropics is also quasi-nondivergent. On the other hand, from a scale analysis of the vorticity and thermodynamic equations, Burger (1991) concluded that \( A = WL/UH \sim 1 \) if the Rossby number is of order unity. This suggests that the upper bound (3.1) may be attained in free synoptic-scale motion in the tropics. In practice it is not helpful to regard synoptic-scale motion in the tropics as free: diabatic effects may determine vertical velocities directly and the upper bound (3.1) may be approached for this reason (see Holton 1972; McBride and Gray 1980; Webster 1983; Hoskins 1987). Synoptic-scale systems of this type are presumably very important in the thermodynamics and dynamics of the tropical atmosphere, and probably of the entire circulation. Even accepting the magnitude of likely errors in estimating the forcing term \( F_j \) in Eq. (2.2), it would appear that retention of \( 2\Omega w \cos \phi \) is necessary for accurate and reliable simulation.

The upper bound (3.1) is also approached in planetary-scale motion. Burger (1991) shows that \( A = WL/UH \sim 1 \) at this scale: hence \( 2\Omega H \cos \phi/U \sim 0.1 \) and retention of the term \( 2\Omega w \cos \phi \) in Eq. (2.2) seems desirable.
The ratio $2\Omega H \cos \phi / U$ (see Eq. (3.2)) occurs if the effects of the height variation of the planetary angular momentum per unit mass, $\Omega r^2 \cos^2 \phi$, are considered (see Eq. (2.8)). The term $2\Omega \omega \cos \phi$ in Eq. (2.2) arises from this height variation. Suppose that a parcel of fluid is initially at rest relative to the earth's surface at latitude $\phi$, and that it then rises a vertical distance $H$. It will acquire a zonal velocity $\Delta U = -2\Omega H \cos \phi$ in the absence of zonal pressure gradients or other zonal forces, if the height variation of the planetary angular momentum is taken into account. Thus, relative to a typical zonal velocity $U$,

$$\left| \frac{\Delta U}{U} \right| \approx 2\Omega H \cos \phi / U.$$  

Dr G. J. Shutts (private communication) has pointed out that values of $\Delta U$ itself indicate the importance of the term $2\Omega \omega \cos \phi$ in Eq. (2.2). At the equator, a 15 km ascent from surface to tropopause will be associated with upper-level easterly flow of about 2 m s$^{-1}$. This effect does not seem small enough to be easily neglected in simulation models, but it is not described by the HPEs. (The effect does not seem to be entirely negligible even in middle latitudes.)

The term $2\Omega \omega \cos \phi$ in Eq. (2.2) may also be assessed in relation to the other Coriolis term, $-2\Omega \nu \sin \phi$. The following argument is due to Professor B. J. Hoskins (private communication). Large-scale motion in the tropics may to a first approximation be described by a Sverdrup-type balance of planetary vorticity advection and vortex stretching (Gill 1980; Hoskins and Karoly 1981; Burger 1991):

$$2\Omega \nu \cos \phi \omega = 2\Omega \sin \phi \left( \frac{\partial \omega}{\partial z} \right).$$

Hence

$$\left| \frac{2\Omega \omega \cos \phi}{2\Omega \nu \sin \phi} \right| \sim \frac{H}{a} \cot^2 \phi.$$  \hspace{1cm} (3.4)

If $H \sim 10^4$ m, $(H/a)\cot^2 \phi$ takes a value of about 0.1 at $\phi = 6^\circ$, while at $\phi = 2^\circ$ it approaches unity. The clear suggestion is that $2\Omega \omega \cos \phi$ cannot comfortably be neglected in Eq. (2.2) when applied to large-scale motion in the tropics. We note that the equatorial Rossby radius of deformation, which is the natural latitudinal scale for equatorially trapped motions, is typically equivalent to 6$^\circ$–12$^\circ$ of latitude.

(b) Scale analysis of the vertical-momentum balance

If $2\Omega \omega \cos \phi$ is retained in Eq. (2.2), then $-2\Omega \nu \cos \phi$ must be retained in Eq. (2.4) in order to preserve consistent energetics. It is nevertheless helpful to carry out a scale analysis of Eq. (2.4), irrespective of energy consistency with Eq. (2.2). The quantity

$$E = \frac{2\Omega U \cos \phi}{g}$$  \hspace{1cm} (3.5)

takes a value of about $1.4 \times 10^{-4} \cos \phi$ (assuming terrestrial values of parameters and $U = 10$ m s$^{-1}$, as in the previous section) and so it might appear that the term $-2\Omega \nu \cos \phi$ is insignificant in Eq. (2.4). But $E$ is not a universal measure of the importance of $-2\Omega \nu \cos \phi$: deviations from a spatial mean hydrostatic balance are much smaller than $g$, and it is these deviations which affect the horizontal motion through the horizontal pressure-gradient terms—see, for example, Holton (1972). If a mean, hydrostatically balanced state is introduced, then the pressure $p$ and density $\rho$ may be expressed as
\[ p = p_0(r) + p' \]
\[ \rho = \rho_0(r) + \rho' \] 
(3.6)

with
\[ \frac{dp_0}{dr} = -\rho_0 g \] 
(3.7)

Eq. (2.4) can then be written as
\[ \frac{Dw}{Dt} - 2\Omega u \cos \phi - \frac{u^2 + v^2}{r} + \frac{g\rho'}{\rho} + \frac{1}{\rho} \frac{\partial p'}{\partial r} = 0 \] 
(3.8)

upon removal of the mean state balance. The importance of \(2\Omega u \cos \phi\) in Eq. (3.8) may be gauged by a comparison with \((1/\rho)(\partial p'/\partial r)\) (which contributes to the deviation hydrostatic balance). We assume tropical scaling, with the Rossby number \(Ro = U/fL \sim 1\). From (2.2), if \(Ro \sim 1\),
\[ \left| \frac{1}{\rho} p' \right| \sim U^2. \]

Hence
\[ \frac{2\Omega u \cos \phi}{U} \sim \frac{2\Omega H \cos \phi}{U} \left| \frac{\partial p'}{\rho \partial r} \right|. \]

The \(\cos \phi\) Coriolis term in Eq. (2.4) is thus negligible compared with the horizontally varying part of the vertical pressure-gradient term only to the extent that
\[ \frac{2\Omega H \cos \phi}{U} \ll 1 \] 
(3.9)

or
\[ E \ll \frac{U^2}{gH} \] 
(3.10)

which are the same as the conditions (3.2) or (3.3)) for neglect of the \(\cos \phi\) Coriolis term in Eq. (2.2). We expect, therefore, that neglect of \(-2\Omega u \cos \phi\) in Eq. (2.4) will lead to errors of up to 10% in the horizontally varying balance which affects the horizontal motion through Eqs. (2.2) and (2.3). (We repeat that this argument applies to tropical balances. In middle latitudes, at least on the synoptic scale, geostrophic control gives larger variations of \(p'\) for a given horizontal velocity-scale \(U\) : \(|p'/\rho| \sim U^2/Ro\), \(Ro \ll 1\). The \(\cos \phi\) Coriolis term in Eq. (2.4) is negligible in this case to the extent that
\[ \frac{2\Omega HRo}{U} \left( \approx \frac{H}{L} \right) \ll 1. \] 
(3.11)

Condition (3.11) is typically well satisfied in middle-latitude motion, for which \(Ro \sim 10^{-1}\) and \(H/L \sim 10^{-2}\).)

It is interesting to note that the derivation of conditions (3.9) and (3.10) does not depend on the upper bound (3.1) to the vertical velocity. Equation (3.1) may be used, however, to show that \(Dw/Dt\) is negligible compared with \(-2\Omega u \cos \phi\) on the synoptic scale:
so

\[ \frac{|Dw/Dt|}{|2\Omega u|} \lesssim \frac{UH}{2\Omega L^2} = \frac{H}{L} \frac{Ro}{10^{-3}} \]

assuming \( L = 10^6 \text{ m}, \ U = 10 \text{ m s}^{-1} \) and \( H = 10^4 \text{ m}\). Draghici (1987, 1989) notes that

\(-2\Omega u \cos \phi \) dominates \( Dw/DT \) for a range of mesoscale motions also, and thus apparently

represents the most important nonhydrostatic effect in such cases.

(c) Previous adiabatic analyses

The above scale analysis suggests that the treatment of dynamically important

balances in Eqs. (2.2) and (2.4) may be subject to errors of about 10\% (at least in the
tropics, or on planetary scales) if the \( \cos \phi \) Coriolis terms are neglected. Phillips (1968)
and Gill (1982) have considered these terms to be less important. From an approximate

dispersion relation for linearized waves in an atmosphere at rest, Phillips identified

\( 4\Omega^2 + N^2 \) (where \( N \) is the buoyancy frequency) as the condition for neglect of the \( \cos \phi \)
Coriolis terms. Gill gave the more stringent condition \( 2\Omega \ll N \) after a scale analysis of

the linearized, equatorial \( \beta \)-plane equations. Either condition reveals an unsatisfactory

aspect of the HPEs: they do not remain a physically acceptable approximation as the

static stability (and hence \( N \)) tends to zero. Nevertheless, if \( N \) is of order \( 10^{-2} \text{ s}^{-1} \) (a
typical tropospheric value) then \( 2\Omega/N \sim 10^{-2} \), and so the \( \cos \phi \) Coriolis terms are

negligible even according to Gill’s criterion.

Both Phillips’s and Gill’s analyses assume adiabatic motion, vertical velocities being

related to horizontal density fluctuations and the buoyancy frequency. In deriving the

condition \( 2\Omega H \cos \phi /U \ll 1 \) for the neglect of the \( \cos \phi \) Coriolis term in Eq. (2.2) (see

Eq. (3.2)), we have estimated vertical velocities from the continuity equation and have

thus used an upper bound which may be approached in regions of strong diabatic heating.

This seems a suitable treatment for tropical synoptic-scale convective complexes in which
diabatic heating plays an important role in the dynamics (including the determination of
phase speeds). Neither Phillips’s nor Gill’s analysis is applicable to planetary-scale

motion.

From the above discussion we conclude that the \( \cos \phi \) Coriolis terms cannot comfort-
ably be neglected. Amongst motions for which they may play a small but not negligible
role are synoptic-scale, diabatically driven flows in the tropics, and planetary-scale flows.
We therefore proceed to examine extensions of the HPEs in which the \( \cos \phi \) Coriolis terms are retained.

4. EXTENSIONS OF THE HYDROSTATIC PRIMITIVE EQUATIONS THAT CONSERVE ENERGY,
ANGULAR MOMENTUM AND POTENTIAL VORTICITY

Some authoritative texts (e.g. Holton 1972; Gill 1982) cite shallow-atmosphere forms
of the components of the Navier–Stokes equations in which the \( \cos \phi \) Coriolis terms and
all of the metric terms are retained. These forms conserve energy but do not imply
precise analogues of angular-momentum and potential-vorticity conservation. Indeed, it
is clear that inclusion of the \( \cos \phi \) Coriolis terms can never be fully consistent with the
shallow-atmosphere approximation: as discussed in section 3(a), the term \( 2\Omega w \cos \phi \) in
Eq. (2.2) represents the height variation of planetary angular momentum, which is
suppressed in the shallow-atmosphere approximation. Since we consider it desirable that

\[ \left| \frac{Dw}{DT} \right| \sim \frac{UW}{L} \leq \frac{U^2 H}{L^2} \]
any proposed extension of the HPEs should possess comparably good conservation laws for angular momentum and potential vorticity as well as energy, we are obliged to abandon the shallow-atmosphere approximation.

(a) A height-coordinate model

Consider an approximation of the Navier–Stokes equation in which the horizontal components are retained unchanged but the material derivative and friction terms are omitted from the vertical component:

\[
\frac{\text{Du}}{\text{Dt}} - \left(2\Omega + \frac{u}{r \cos \phi}\right)(v \sin \phi - w \cos \phi) + \frac{1}{\rho r \cos \phi} \frac{\partial p}{\partial \lambda} = F_{\lambda},
\]

(4.1)

\[
\frac{\text{Dv}}{\text{Dt}} + \left(2\Omega + \frac{u}{r \cos \phi}\right)u \sin \phi + \frac{v w}{r} + \frac{1}{\rho r} \frac{\partial p}{\partial \phi} = F_{\phi},
\]

(4.2)

\[- \left(2\Omega + \frac{u}{r \cos \phi}\right)u \cos \phi - \frac{v^2}{r} + g + \frac{1}{\rho} \frac{\partial p}{\partial r} = 0,
\]

(4.3)

where \(\text{D}/\text{Dt}\) is given by Eq. (2.5).

Equations (4.1)–(4.3) may be written in vector form as

\[
\frac{\text{Du}}{\text{Dt}} - \mathbf{k} \frac{\text{Dw}}{\text{Dt}} + 2\Omega \times \mathbf{u} + g\mathbf{k} + \frac{1}{\rho} \text{grad} p = \mathbf{F}_h,
\]

(4.4)
in which \(\mathbf{F}_h = (F_{\lambda}, F_{\phi}, 0)\) represents the horizontal components of the external force per unit mass. Equation (4.4) implies conservation of axial angular momentum because its zonal component (4.1) is the unapproximated Navier–Stokes form.

The unapproximated continuity and thermodynamic equations (Eqs. (2.6) and (2.7)), taken together with Eq. (4.4), imply the energy-conservation law

\[
\rho \frac{\text{D}}{\text{Dt}} \left(\frac{1}{2} \mathbf{v}^2 + \Phi + c_i T\right) + \text{div}(\rho \mathbf{u}) = \rho(Q + \mathbf{v} \cdot \mathbf{F}_h)
\]

(4.5)

which differs from the unapproximated form, Eq. (2.9), only in the absence of the contribution of \(\frac{1}{2}w^2\) to the specific kinetic energy and of the contribution of \(\rho w F\), to the rate of working of the external force \(\mathbf{F}\).

The potential-vorticity properties of the model (Eqs. (4.4), (2.6) and (2.7)) may be established by noting that

\[
\text{curl} \left(\mathbf{k} \frac{\text{Dw}}{\text{Dt}}\right) = \frac{\text{D}\xi}{\text{Dt}} + \xi \text{div} \mathbf{u} - (\xi \cdot \text{grad})\mathbf{u}
\]

(4.6)

where

\[
\mathbf{r}\xi = i \frac{\partial w}{\partial \phi} - j \frac{1}{\cos \phi} \frac{\partial w}{\partial \lambda}
\]

(4.7)

(i and j are unit vectors in the zonal and meridional directions.) Equation (4.6) is a consequence of the special symmetry of the spherical polar-coordinate system; a proof is outlined in White and Bromley (1994).

Using Eqs. (4.6), (4.7) and the known properties of the Navier–Stokes equation it follows that Eq. (4.4) implies the vorticity equation

\[
\frac{\text{DZ'}}{\text{Dt}} + Z' \text{div} \mathbf{u} - (Z' \cdot \text{grad})\mathbf{u} + \text{curl} \left(\frac{1}{\rho} \text{grad} p\right) = \text{curl} \mathbf{F}_h.
\]

(4.8)
Here $Z' = Z - \xi$ is the absolute vorticity (cf. Eq. (2.11)) stripped of all terms involving $w$:

\[
\begin{align*}
Z'_1 &= -\frac{1}{r} \frac{\partial}{\partial r} (rv) \\
Z'_\phi &= 2\Omega \cos \phi + \frac{1}{r} \frac{\partial}{\partial r} (ru) \\
Z'_z &= 2\Omega \sin \phi + \frac{1}{r \cos \phi} \left( \frac{\partial u}{\partial \lambda} - \frac{\partial}{\partial \phi} (u \cos \phi) \right).
\end{align*}
\]

(4.9)

If the motion is frictionless and adiabatic it then follows from Eqs. (2.6), (2.7) and (4.8) that

\[
\frac{D}{Dt} \left( \frac{Z' \cdot \text{grad} \theta}{\rho} \right) = 0.
\]

(4.10)

Thus Eqs. (4.4), (2.6) and (2.7) imply an analogue of Ertel's potential-vorticity theorem.

Miller and Gall (1983) used a form similar to (4.4) in a numerical study of zonally symmetric motion of a Boussinesq fluid. They noted its consistent energetics but did not discuss the angular momentum or potential-vorticity properties.

(b) A pressure-coordinate model

Since Eq. (4.3) is a nonhydrostatic form, it might be expected that pressure-coordinate transforms of the equations considered in section 4(a) would not be useful. However, the hydrostatic approximation remains an accurate statement except where horizontal variations of the balance represented by (4.3) are relevant, and progress can be made using the technique applied by Miller (1974) to develop a pressure-coordinate model of nonhydrostatic convective motion. As well as retaining all metric terms, we also avoid the shallow-atmosphere approximation. Instead, $r$ is replaced by the pseudo-radius $r_s(p)$ defined as

\[
r_s(p) = a + \int_p^{p_1} \frac{RT_s(p')}{gp'} \, dp'.
\]

(4.11)

Hence $r_s(p)$ is the mean radius of the earth plus the height $z_s(p)$ defined by Eq. (2.33). $T_s(p)$ is to be interpreted here as a profile representing the horizontally averaged, hydrostatically balanced state of the atmosphere. According to Eq. (4.11)

\[
\frac{Dr_s}{Dt} = - \frac{RT_s(p)\omega}{gp} = \tilde{w}
\]

(4.12)

which is a key element in the dynamical consistency of the new model. In (4.12), and below,

\[
\frac{D}{Dt} = \frac{\partial}{\partial t} + \frac{u}{r_s \cos \phi} \frac{\partial}{\partial \lambda} + \frac{v}{r_s} \frac{\partial}{\partial \phi} + \frac{\omega}{\partial p}
\]

(4.13)

all differentiations with respect to $t$, $\lambda$ and $\phi$ being taken at constant $p$.

The proposed $p$-coordinate equations are:

\[
\frac{Du}{Dt} - \left(2\Omega + \frac{u}{r_s \cos \phi}\right)(v \sin \phi - \tilde{w} \cos \phi) + \frac{1}{r_s \cos \phi} \frac{\partial \Phi}{\partial \lambda} = F_\lambda
\]

(4.14)
In the vertical-component equation (4.16),
\[ \mu = (2\Omega u_r \cos \phi + u^2 + v^2)/r_s g \tag{4.19} \]
which is an extension of the quantity \( E \) introduced in section 3 (see Eq. (3.5)). In the continuity equation (4.17),
\[ \vec{\nabla}_p \cdot \mathbf{v} = \frac{1}{r_s \cos \phi} \left( \frac{\partial u}{\partial \lambda} + \frac{\partial}{\partial \phi} (v \cos \phi) \right). \tag{4.20} \]

The quantity \( \mu \) is dominated by the contribution from the term \( 2\Omega u \cos \phi \) whose inclusion is the aim of this formulation. In Eqs. (4.14)–(4.18) \( \mu \) only appears in (4.16), the vertical-component equation. The mean-state balance may be extracted from (4.16) to leave the nonhydrostatic form
\[ \frac{RT}{p} + \frac{\partial \Phi}{\partial p} + \mu \frac{RT_s}{p} = 0 \tag{4.21} \]
where primes indicate departures from the mean state. If the height-coordinate forms (2.6), (2.7), (4.1) and (4.2) are transformed to pressure coordinates, terms in \( p \) appear also in the horizontal-momentum components and in the continuity equation. These terms are neglected according to the smallness of \( p \).

The approximations \( w = \frac{\omega}{g} = -\frac{RT_s \omega}{g p} = \tilde{w} \) have been used to express in convenient forms the terms in \( w \) which appear in the \( \cos \phi \) Coriolis terms and the metric terms of Eqs. (4.1) and (4.2). The minute terms \( u \tilde{w}/r_s \) and \( v \tilde{w}/r_s \) are retained in Eqs. (4.14) and (4.15) solely to ensure the conservation properties of the system (see below). In all other respects the treatment is precisely analogous to that applied by Miller (1974). Miller and White (1984) and White (1989) have offered further discussion and justification of Miller’s technique; see also Salmon and Smith (1994). The scale analysis for the present case is laid out in detail in White and Bromley (1994).

By using (4.12) it is easily seen that the zonal component (4.14) implies the axial angular-momentum principle
\[ \frac{D}{Dt} \left( (u + \Omega r_s \cos \phi) r_s \cos \phi \right) = F_\phi r_s \cos \phi - \frac{\partial \Phi}{\partial \lambda}. \tag{4.22} \]
This is evidently a good analogue of the Navier–Stokes form (Eq. (2.8)), though the omission of the very small term \(-\left( \mu/r_s \cos \phi \right)(\partial \Phi/\partial \lambda)\) during the derivation of (4.14) from (4.1) should be noted.
The energy equation implied by Eqs. (4.14)-(4.18) is

$$\frac{D}{Dt} \left( \frac{1}{2} v^2 + c_p T \right) + \nabla \cdot (v \Phi) + \frac{1}{r_s^2} \frac{\partial}{\partial p} (r_s^2 \omega \Phi) = Q + v \cdot \mathbf{F}_h$$  \hspace{1cm} (4.23)

which is reminiscent of the HPE form (2.39).

The potential-vorticity properties of the new model may be established by noting an isomorphism to the height-coordinate model described in section 4(a). From (4.11) and (4.12) it follows that

$$\frac{\partial}{\partial p} = - \frac{RT_s}{g p} \frac{\partial}{\partial r_s}$$

and

$$\omega \frac{\partial}{\partial p} = \overline{w} \frac{\partial}{\partial r_s}.$$

Hence

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \frac{u}{r_s \cos \phi} \frac{\partial}{\partial \lambda} + \frac{v}{r_s} \frac{\partial}{\partial \phi} + \overline{w} \frac{\partial}{\partial r_s}$$  \hspace{1cm} (4.24)

(cf. (2.5)), and the vertical component (4.16) may be written as

$$-2 \Omega u \cos \phi - \frac{u^2 + v^2}{r_s} - \frac{gT}{T_s} \frac{\partial \Phi}{\partial r_s} = 0.$$  \hspace{1cm} (4.25)

The inertial, metric and Coriolis terms in Eqs. (4.14)-(4.16) are thus isomorphic to the corresponding terms in the height-coordinate forms (Eqs. (4.1)-(4.3)). Further, the isomorphism extends to the commutation properties of the relevant operators. For frictionless, adiabatic flow Eqs. (4.14)-(4.18) imply the potential-vorticity conservation law

$$\frac{D}{Dt} \left( \frac{\mathbf{Z} \cdot \nabla \theta}{\rho_s} \right) = 0.$$  \hspace{1cm} (4.26)

Here $\rho_s = p/RT_s$

$$\nabla \equiv \left( \frac{1}{r_s \cos \phi} \frac{\partial}{\partial \lambda}, \frac{1}{r_s} \frac{\partial}{\partial \phi}, \frac{\partial}{\partial r_s} \right)$$  \hspace{1cm} (4.27)

$$\mathbf{Z} = \left\{ -\frac{1}{r_s} \frac{\partial}{\partial r_s} (ur_s), 2 \Omega \cos \phi + \frac{1}{r_s} \frac{\partial}{\partial r_s} (ur_s), 2 \Omega \sin \phi + \frac{1}{r_s \cos \phi} \left( \frac{\partial v}{\partial \lambda} - \frac{\partial}{\partial \phi} (u \cos \phi) \right) \right\}.$$  \hspace{1cm} (4.28)

Vertically propagating acoustic modes are not implied by the new system. This follows from either the omission of $Dw/Dt$ in the vertical-component equation or the isomorphism of the continuity equation (4.17) to an incompressible fluid form. Horizontally propagating acoustic modes—the Lamb modes—will be present (as in the HPEs) unless the boundary condition $\omega = 0$ is applied on appropriate pressure levels (see, for example, Miller and White (1984)).
(c) \( \sigma \)-coordinate forms

Transformation of the pressure-coordinate equations (4.14)-(4.19) to \( \sigma \)-coordinates \((\sigma = p/p_*, p_* = \text{surface pressure})\) gives:

\[
\frac{Du}{Dt} - \left(2\Omega + \frac{u}{r_s \cos \phi}\right)(v \sin \phi - w \cos \phi) +
\frac{1}{r_s \cos \phi} \left(\frac{\partial \Phi}{\partial \lambda} + \frac{R(T + \mu T_*) \partial p_*}{p_*} \frac{\partial \phi}{\partial \lambda}\right) = F_l
\]

\[
\frac{Dv}{Dt} + \left(2\Omega + \frac{u}{r_s \cos \phi}\right)u \sin \phi + \frac{v \tilde{w}}{r_s} + \frac{1}{r_s} \left(\frac{\partial \Phi}{\partial \phi} + \frac{R(T + \mu T_*) \partial p_*}{p_*} \frac{\partial \phi}{\partial \phi}\right) = F_\phi
\]

\[
\frac{\partial \Phi}{\partial \sigma} + \frac{R}{\sigma} (T + \mu T_*) = 0
\]

\[
\frac{\partial p_*/ \partial \sigma}{\partial \sigma} (\sigma r_*^2) + r_*^2 \nabla_{\sigma} \cdot (p_* v) + p_* \frac{\partial}{\partial \sigma} (r_*^2 \dot{\sigma}) = 0
\]

\[
c_p \frac{DT}{Dt} - \frac{RT \omega}{p} = Q.
\]

All differentiations with respect to \( t, \lambda \) and \( \phi \) are carried out at constant \( \sigma \),

\[
\frac{D}{Dt} = \frac{\partial}{\partial t} + \frac{u}{r_s \cos \phi} \frac{\partial}{\partial \lambda} + \frac{v}{r_s} \frac{\partial}{\partial \phi} + \frac{\sigma}{\partial \sigma} \frac{\partial}{\partial \sigma}
\]

and

\[
\nabla_{\sigma} \cdot (p_* v) = \frac{1}{r_*^2 \cos \phi} \left(\frac{\partial}{\partial \lambda} (p_* u r_s) + \frac{\partial}{\partial \phi} (p_* v r_s \cos \phi)\right).
\]

The quantity \( \mu \) which appears in Eqs. (4.29)-(4.31) is defined by Eq. (4.19). The terms in \( \mu T_* \) in Eqs. (4.29) and (4.30) are formally negligible compared with the accompanying terms in \( T \), but they must be retained in order to preserve the conservation properties of the formulation. \( r_s = r_s(p) \) is defined by Eq. (4.11) (and so multiplication by \( r_s(p) \) does not commute with the operators \( \partial/\partial t, \partial/\partial \lambda, \partial/\partial \phi \) taken at constant \( \sigma \); hence the form of the divergence operator in (4.35)). \( \tilde{w} \) is defined by Eq. (4.12), with

\[
\frac{\omega}{p} = \left(\frac{\dot{\sigma} + \frac{1}{p_*} \frac{Dp_*}{Dt}}{\sigma} \right).
\]

Numerical time integration of (4.29)-(4.33) may be carried out essentially as for the HPEs. The continuity equation (4.32) is used in the integrated forms

\[
r_*^2(p_*) \frac{\partial p_*/ \partial t}{\partial t} = -\int_0^1 r_*^2 \nabla_{\sigma} \cdot (p_* v) \, d\sigma
\]

\[
\dot{\sigma} = -\frac{1}{p_*} \frac{\partial p}{\partial t} - \frac{1}{p_* r_*^2} \int_0^\sigma r_*^2 \nabla_{\sigma} \cdot (p_* v) \, d\sigma
\]

to find \( \partial p_*/\partial t \) and then \( \dot{\sigma} \). The vertical component (4.31) is then used to find \( \Phi \) (knowing \( u, v \) and \( T \) from the previous time step) in the same way as the hydrostatic relation is used to find \( \Phi \) from \( T \) (and \( \Phi_* \)) when integrating the HPEs.
5. Discussion

In this paper we have examined the importance of the $\cos \phi$ Coriolis terms which appear in the zonal and vertical components of the momentum equation. These terms are unimportant in synoptic-scale, quasi-adiabatic motion in middle latitudes: they attain magnitudes of only 1%, or less, of the Lagrangian rate of change in the zonal-component equation and of the horizontally varying part of the pressure-gradient term in the vertical-component equation. In the tropics, where the Rossby number may be of order unity and vertical velocities may be determined by diabatic heating even on the synoptic scale, the $\cos \phi$ Coriolis terms may attain magnitudes of about 10%, in the above senses, in both the zonal- and vertical-component equations. Similar magnitudes may also be approached in middle latitudes in frontal zones—where diabatic heating may be large and isentropic slopes steep (Draghici 1987, 1989). In planetary-scale motion the $\cos \phi$ Coriolis term in the zonal component of the momentum equation may attain a magnitude of about 10% of the Lagrangian rate-of-change term.

Terms attaining such magnitudes might reasonably be considered negligible in theoretical treatments and models aimed at developing conceptual understanding of atmospheric behaviour. But it seems inconsistent with the rationale of numerical flow simulation to omit known terms that attain these magnitudes. In view of the sophistication of modern weather-prediction and climate-simulation models, we consider that the $\cos \phi$ Coriolis terms should now be retained in such models. Certainly, scale analysis suggests that their retention is more comfortable than their omission.

The hydrostatic primitive equations do not include the $\cos \phi$ Coriolis terms, but they imply good conservation properties. Indeed, the theoretical acceptability of the HPEs probably rests as much on the existence and nature of the conservation laws ((2.21)–(2.23)) as on a conviction that the omitted terms are negligible. The HPEs set a demanding standard against which any proposed extended forms should be judged. If the conservation laws are to be fully respected, the $\cos \phi$ Coriolis terms cannot be included in the HPEs without making other changes to the equations at the same time.

In section 4 we have proposed an extended version of the HPEs in which the $\cos \phi$ Coriolis terms are included within a global, quasi-hydrostatic, acoustically filtered formulation. A limited relaxation of the shallow-atmosphere approximation is incorporated. The model conserves energy, axial angular momentum and potential vorticity. It has been presented in pressure and $\sigma$-coordinate forms. This new model is put forward as a more accurate alternative to the traditional HPEs: it has comparably good conservation properties and is as easy to integrate numerically using grid-point methods. This formulation, written in an appropriate vertical-coordinate system, is used in the Meteorological Office's Unified Model for weather prediction and climate simulation (Cullen 1993).

Acknowledgements

This paper has presented ideas which have been many years in gestation. We are grateful to the colleagues who have advised and encouraged us during this time, among them Professor R. Hide and Drs M. J. P. Cullen, G. J. Shutts and I. Roulstone. The benefit of many useful discussions with Professor B. J. Hoskins is particularly appreciated. The comments of an anonymous referee have also been helpful.
APPENDIX

List of symbols and definitions

\(a\) Earth’s mean radius
\(c_v\) Specific heat at constant volume
\(c_p\) Specific heat at constant pressure
\(f\) Coriolis parameter: \(2\Omega \sin \phi\)
\(g\) Acceleration due to apparent gravity
\(g\) Magnitude of \(g\)
\(i\) Unit vector in zonal direction
\(j\) Unit vector in meridional direction
\(k\) Unit vector in direction of apparent vertical (direction of \(-g\))
\(p\) Pressure
\(p_0\) A hydrostatically balanced reference pressure profile
\(p_*\) Surface pressure
\(p_1\) A constant reference surface pressure
\(r\) Position vector relative to the centre of the earth
\(r\) Distance from the centre of the earth
\(t\) Time
\(u\) Velocity (relative to earth)
\(u\) Zonal component of \(u\)
\(v\) Horizontal part of \(u\) (=\((u, v, 0)\))
\(v\) Meridional component of \(u\)
\(w\) Vertical component of \(u\)
\(\tilde{w}\) Approximation to \(w\) (see (2.36) and (4.12))
\(z\) Height above mean sea level
\(E\) \(2\Omega U \cos \phi / g\)
\(F = (F_x, F_y, F_z)\) Frictional force per unit mass
\(F_h\) Horizontal part of \(F = (F_x, F_y, 0)\)
\(H\) Vertical space scale
\(L\) Horizontal space scale
\(N\) Buoyancy frequency: \(\{(g/\theta) \, d\theta/dz\}^{1/2}\)
\(Q\) Diabatic heating rate per unit mass
\(R\) Gas constant per unit mass
\(Ro\) Rossby number: \(U/fL\)
\(T\) Temperature
\(U\) Horizontal velocity scale
\(W\) Vertical velocity scale
\(Z = (Z_x, Z_y, Z_z)\) Absolute vorticity (see (2.11))
\(\zeta\) Absolute vorticity in HPE models (see (2.25))
\(\theta\) Potential temperature: \(T(p_1/p)^{R/c_p}\)
\(\lambda\) Longitude
\(\mu\) Ratio defined in (4.19)
\(\rho\) Density
\(\rho_0\) A hydrostatically balanced reference density profile
\(\phi\) Latitude
\(\sigma\) \(p/p_*\)
\(\dot{\sigma}\) \(Da/Dr\)
\(\Phi\) Geopotential
\(\Omega\) Angular velocity of earth’s rotation
EQUATIONS FOR GLOBAL MODELS

\( \Omega \)  Magnitude of \( \Omega \)
\( \nabla \)  Shallow-atmosphere gradient operator

REFERENCES

Eckart, C.  1960  *The hydrodynamics of oceans and atmospheres*. Pergamon Press
Lorenz, E. N.  1967  The nature and theory of the general circulation of the atmosphere. WMO No. 218, TP 115
Pedlosky, J.  1987  *Geophysical fluid dynamics*. Springer-Verlag


