1. Droplet Growth by Condensation

It was shown before that a critical size $r^*$ and saturation ratio $S^*$ must be exceeded for a small solution droplet to become a cloud droplet. Before the droplet reaches the critical size, it grows by diffusion of water molecules from the vapor onto its surface. The rate of diffusional growth of a single droplet is analyzed in this section. The droplet has radius $r$ and is located in a vapor field with the concentration of vapor molecules at distance $R$ from the droplet center denoted by $n(R)$. We can characterize the vapor field in terms of $\rho_v(R) = n(R)m_0$ where $m_0$ denotes the mass of one water molecule. We assume isotropy so that $n(R)$ doesn’t depend on direction. At any point in the vapor field, the concentration of molecules is assumed to satisfy the diffusion equation:

$$\frac{\partial n}{\partial t} = D\Delta^2 n$$  \hspace{1cm} (1)

Using the divergence in polar coordinates, and obtaining the expression for steady-state conditions:

$$D\Delta^2 n(R) = 0 = 1 \frac{\partial}{R^2} \left( R^2 \frac{\partial n}{\partial R} \right)$$  \hspace{1cm} (2)

Which has a general solution $n(R) = C_1 - C_2/R$ with boundary conditions:

- $R \to \infty$ and $n \to sn_\infty$, the ambient or undisturbed value of vapor concentration,
- $R \to r$ and $n \to n_r$, the vapor concentration at the droplet’s surface.

We get the solution:

$$n(R) = n_\infty - \frac{r}{R}(n_\infty - n_r)$$  \hspace{1cm} (3)

1a. Growth of mass by Condensation

The flux of molecules onto the surface of the droplet is $D(\partial n/\partial R)_{R=r}$, so the rate of mass increase is:

$$\frac{dm}{dt} = 4\pi r^2 D \left( \frac{\partial n}{\partial R} \right)_{R=r} m_0$$  \hspace{1cm} (4)

So we can obtain the change of mass in time as:
\[
\frac{dm}{dt} = 4\pi r D (n_\infty - n_r) m_0 \tag{5}
\]

\[
\frac{dm}{dt} = 4\pi r D (\rho_v - \rho_{vr}) \tag{6}
\]

were \( \rho_v \) is the ambient vapor density and \( \rho_{vr} \) is the vapor density at the droplet’s surface.

**1b. Latent Heat release**

Associated with condensation is the release of latent heat which raises the droplet temperature above the ambient value. The diffusion of heat away from the droplet is given by

\[
\frac{dQ}{dt} = 4\pi r K (T_r - T) \tag{7}
\]

where \( T \) is the ambient temperature and \( T_r \) is the temperature at the surface of the droplet, and \( K \) is the coefficient of thermal conductivity of the air.

Condensation of water vapor releases heat \( ml \) and the rate of release is \( l \frac{dm}{dt} \). The latent heat of condensation is transported by diffusion away from the droplet (assuming no changes in temperature with time - steady state). We can then derive the following equation:

\[
\frac{dQ}{dt} = l \frac{dm}{dt} \tag{8}
\]

\[
\frac{\rho_v - \rho_{vr}}{T_r - T} = K \frac{Dl}{m} \tag{9}
\]

Here \( T_r \) and \( \rho_{vr} \) are unknown, but we know the expression for the equilibrium pressure over a droplet:

\[
\rho_{vr} = \frac{\varepsilon_v(r)}{R_v T_r} = \frac{\varepsilon_v(T_r)}{R_v T_r} \left( 1 + \frac{a}{r} - \frac{b}{r^3} \right) \tag{10}
\]

These two equations can be solved simultaneously for \( T_r \) and \( \rho_{vr} \), and then solve for the growth. However, there is a simplified expression for the growth:
\[
\frac{dr}{dt} = \frac{(S - 1) - \frac{a}{r} + \frac{b}{r^3}}{F_k + F_d}
\]  

(11)

where \( S = \frac{e}{e_s} \) is the ambient saturation ratio. \( F_k = \left( \frac{L}{R_v T} - 1 \right) \frac{l}{K_T} \), is the thermodynamic-heat conduction term, and \( F_d = \frac{\rho R_v T}{D_v(T)} \) is the vapor diffusion term. The values for \( K \) and \( D \) vary with temperature (See Fig 1).

For very large radii:

\[
\frac{dr}{dt} = \frac{(S - 1)}{F_k + F_d}
\]  

(12)

Integrating, we get:

\[
r^2(t) = r^2(t_0) + \frac{2(S - 1)}{F_k + F_d} (t - t_0)
\]  

(13)

This leads to a narrowing of drop size distribution with time. Initially, droplets that form on large condensation nucleus grow faster, but after a certain radius they homogenize. We can also use the equation for evaporation and discriminate between cloud and rain droplets.

The formation of cloud droplets proceeds as follows:
- Adiabatic ascent enables cooling and eventually saturation with respect to a plane surface of liquid.
- Further ascent produces supersaturation.
- CCNs are activated, starting with the most efficient.
- When the rate at which water vapor in excess of saturation is equal to the rate at which vapor condenses onto the CCN and droplets, supersaturation reaches a maximum.
- Concentration of droplets is determined at this stage.
- Subsequently, growing droplets consume water vapor faster than it is made available by cooling of air, so supersaturation decreases.
- Haze droplets begin to evaporate, activated droplets continue to grow.
- If the only process occurring were condensation, the droplets would become increasingly uniform with time to about 10 \( \mu m \).
- Condensation alone in warm clouds is much too slow to produce raindrops with radii of several millimeters.
- Yet, rain does form in warm clouds.
- To go from 100\( \mu m \) to 1 mm requires an increase in volume of one millionfold!
However, only one droplet in a million in a cloud has to grow by this amount for the cloud to rain.

The mechanism responsible for the selective growth of a few droplets into raindrops in warm clouds is Collision and Coalescence.
The question is, how can we get these larger (20\(\mu m\)) droplets that can start the collection process.

1. **Giant cloud condensation nuclei.**
2. **Turbulence** can produce fluctuating supersaturations that enhance condensational growth and updrafts and downdrafts.
3. **Radiative broadening.** When a droplet is growing it is warmer than environment, so the droplet can lose heat by radiation so the saturation vapor pressure will be lower.
4. **Stochastic collection.** Accounts for the fact that collisions are stochastic events, distributed statistically in space and time. For example, if at every time step, only 90% of the droplets collide with other droplets, you can quickly get a broad spectrum of size distributions.

### 2. Collision and Coalescence

Collisions are influenced by gravitational, electrical and aerodynamic forces. Gravitational effects dominate in clouds where large droplets capture small ones.

collision efficiency \( E_{\text{coll}} = (\# \text{ collisions}) / (\# \text{ encounters in geometric sweep}) \)

coalescence efficiency \( E_{\text{coal}} = (\# \text{ coalescences}) / (\# \text{ collisions}) \)

collection efficiency \( E = E_{\text{coll}} \times E_{\text{coal}} \)

Theoretical studies generally assume that the \( E = E_{\text{coll}} \), for a coalescence efficiency of 1.

#### 2a. Droplet terminal fall speed

i. **The drag force** on a sphere of radius \( r \) is:

\[
F_R = \frac{\pi}{2} r^2 u^2 \rho C_D
\]

(14)

\( u \) is the velocity of the sphere relative to the fluid. \( \rho \) is the fluid density and \( C_D \) is the drag coefficient. In terms of the Reynolds number \( Re = \frac{2pur}{\mu} \), where \( \mu \) is the
dynamic viscosity. The Reynolds number is a non-dimensional ratio of inertial to viscous forces.

\[ F_R = \frac{\pi \mu ru C_D Re}{4} \tag{15} \]

ii. The gravitational force minus buoyancy on a sphere of radius \( r \) is:

\[ F_A = \frac{4}{3} \pi r^3 g (\rho_l - \rho) \tag{16} \]

iii. Terminal fall speed occurs when \( F_A = F_R \)

\[ \frac{4}{3} \pi r^3 g (\rho_l - \rho) = \frac{\pi \mu ru C_D Re}{4} \tag{17} \]

if \( \rho_l \gg \rho \)

\[ u^2 = \frac{8rg\rho_l}{3\rho C_D} \tag{18} \]

\[ u = \frac{16r^2g\rho_l}{3\mu C_D Re} \tag{19} \]

For very small Reynolds numbers \( \frac{C_D Re}{24} = 1 \)

\[ u = \frac{2r^2g\rho_l}{9\mu} \tag{20} \]

For droplets \( \approx 30 \mu m \), \( u = k_1 r^2 \). This quadratic dependence of fall speed on size is called Stoke’s Law. \( k_1 = 1.19 \times 10^6 \text{s}^{-1} \text{cm}^{-1} \).

For high Reynolds numbers, \( C_D \) is independent of \( Re \), and \( C_D = 0.45 \).

\( u = k_2 r^{1/2} \) and \( k_2 = 2.2 \times 10^3 \left( \frac{\rho_0}{\rho} \right)^{1/2} \text{cm}^{1/2} \text{s}^{-1} \). Where \( \rho \) is the air density and \( \rho_0 \) is a reference density of \( 1.2 \text{kg/m}^3 \). The region in between has a linear relation \( u = k_3 r \) where \( k_3 = 8 \times 10^3 \text{s}^{-1} \).
2b. **Collision Efficiency**

- A drop of radius $R$ overtaking a drop of radius $r$.
- An object with zero inertia would be swept aside.
- The relative importance of the balance between inertial and aerodynamic forces and the separation $x$ between drop centers.
- At a critical distance $x_0$ the droplet makes a grazing collision. At $x < x_0$ the droplet collides.

The effective collision cross section of the collector drop of radius $R$ is $[\pi x_0^2]$. The geometric collision cross section $[\pi (R + r)^2]$

$$E_{\text{coll}}(R, r) = \frac{x_0^2}{(R + r)^2}$$  \hspace{1cm} (21)

if $x_0 < (r + R)$ not all drops in the geometrical collision cross section will collide with the collector drop. These calculations are mathematically involved.

- Collision efficiency is small for small $r/R$
- Increases as $r/R$ increases up to 0.6
- At around 0.6, the radii are similar and relative velocities are small, the flow fields interact strongly
- at $r/R$ close to 1, there is a ”wake effect” and the efficiency can be greater than one.

2c. **Growth Equations**

The drop of radius $R$ is falling at terminal speed through a population of smaller droplets. the geometric collision volume swept is calculated as:

$$V_{\text{coll}} = \pi (R + r)^2 [U(R) - u(r)] \Delta t$$  \hspace{1cm} (22)

The average number of droplets with radii between $r$ and $dr$ is $n(r)dr$, so the average number of droplets collected in unit time is:

$$N_{\text{coll}} = \pi (R + r)^2 [U(R) - u(r)] E(R, r) n(r) dr$$  \hspace{1cm} (23)
FIG. 8.3. Field of $E(R, r)$ based on data in Table 8.2.
for $r = 100\mu m$ assume coalescence efficiency =1, so the collection and collision efficiencies are equal.

The rate of increased volume with time is calculated as the number of droplets times the volume of droplets integrated over all droplet sizes.

$$\frac{dV}{dt} = \int_0^R \frac{4}{3} \pi r^3 \pi (R + r)^2 [U(R) - u(r)] E(R, r)n(r)dr$$  \hspace{1cm} (24)

We can express it in terms of the radius of the collector drop:

$$\frac{dR}{dt} = \frac{\pi}{3} \int_0^R \frac{(R + r)^2}{R^2} [U(R) - u(r)] n(r)r^3 Edr$$  \hspace{1cm} (25)

If the droplets are much smaller than the collector drop $u(r) \approx 0$ and $(R+r) \approx R$

$$\frac{dR}{dt} = \frac{U(R)\bar{E}M}{4\rho_l}$$  \hspace{1cm} (26)

Where $M = V_l\rho_l$ is the cloud liquid water and $V_l = \frac{M}{\rho_l}$, $\bar{E}$ is the effective average of collision efficiency for droplet population.

$U(R)$ increases as $R$ increases, $dR/dt$ increases with increasing $R$. For this reason, at the beginning condensation dominates, but after a certain threshold, growth by collisions dominates.

2d. Change of drop size with Altitude

The relative speed of the droplet determines the distance traveled:

$$\frac{dz}{dt} = w - U(R)$$  \hspace{1cm} (27)

Where $w$ is the updraft speed and $U(R)$ is the fall speed of the drop.

$$\frac{dR}{dz} = \frac{\bar{E}MU}{4\rho_l(w - U(R))}$$  \hspace{1cm} (28)

If $w \ll U(R)$ then $\frac{dn}{dz} = -\frac{\bar{E}M}{4\rho_l}$ These equations describe droplet growth as
a continuous collection process. We can also use it to determine the value of $H$ (height) for any radius $R_H$. If the radius of the collector drop and any height $H$ is $R_H$ and the radius at cloud base is $R_0$, If we assume $M$ is independent of $z$ (homogeneous cloud liquid water) - then

$$H = \frac{4\rho_l}{M} \left[ \int_{R_0}^{R_H} \frac{w}{U(R)\epsilon} dR - \int_{R_0}^{R_H} \frac{dR}{\epsilon} \right]$$

(29)

- For small droplets $w > U(R)$ and the first integral dominates and $H$ increases as $R_H$ increases so the drop growing by collision is carried upward.

- For $w < U(R)$, $H$ decreases with increasing $R_H$ and the drop begins to fall. Eventually passing through cloud base and reaching the ground if it doesn’t evaporate along the way.

3. Bowen Model, 1950

He assumed that a cloud of uniformly sized droplets was ascending with a constant updraft and growing by condensation. Then one drop with twice the mass was placed to grow by collision. Showed that growth is slow at first, then coalescence outpaces condensation. When it has grown to a size that will just be balanced by the updraft, the drop reaches the top of the trajectory. Then it descends and emerges as a raindrop.

With increasing updraft velocity, the drop ascends to a higher level before beginning its descent, then emerges from cloud base with a larger size. Drops grow larger but have a lower trajectory as the cloud water content is increased. In Figures 8.5 and 8.6 one droplet of $20\mu$ is placed in a population of $10\mu$ droplets. The most serious discrepancy between model predictions and observations is the time requirement. While calculations indicate that about an hour is required to produce millimeter-sized drops, observations show that such drops can form in 15-30 minutes. Statistical growth is key to this physical process.
FIG. 8.4. Bowen's calculated trajectories of (a) the air, (b) cloud droplets, initially 10 μm in radius, and (c) drops which have initially twice the mass of the cloud droplets. Updraft speed 1 m/sec, cloud water content $M = 1$ g/m$^3$. (From Fletcher, 1962.)
FIG. 8.5. Drop trajectories calculated for the collision efficiencies of Table 8.2 and Fig. 8.3, assuming a coalescence efficiency of unity. Initial drop radius 20\(\mu\)m. Cloud water content 1 g/m\(^3\); all cloud droplets of 10\(\mu\)m radius.

FIG. 8.6. Drop diameters for the trajectories of Fig. 8.5.