1. Droplet Growth by Condensation

It was shown before that a critical size $r*$ and saturation ratio $S*$ must be exceeded for a small solution droplet to become a cloud droplet. Before the droplet reaches the critical size, it grows by diffusion of water molecules from the vapor onto its surface. The rate of diffusional growth of a single droplet is analyzed in this section. The droplet has radius r and is located in a vapor field with the concentration of vapor molecules at distance R from the droplet center denoted by $n(R)$. We can characterize the vapor field in terms of $\rho_v(R) = n(R)m_0$ where m_0 denotes the mass of one water molecule. We assume isotropy so that $n(R)$ doesn't depend on direction. At any point in the vapor field, the concentration of molecules is assumed to satisfy the diffusion equation:

$$\frac{\partial n}{\partial t} = D\Delta^2 n \tag{1}$$

Using the divergence in polar coordinates, and obtaining the expression for steady-state conditions:

$$D\Delta^2 n(R) = 0 = \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial n}{\partial R}\right) \tag{2}$$

Which has a general solution $n(R) = C_1 - C_2/R$ with boundary conditions:

- $R \to \infty$ and $n \to sn_\infty$, the ambient or undisturbed value of vapor concentration,
- $R \to r$ and $n \to n_r$, the vapor concentration at the droplet's surface.

We get the solution:

$$n(R) = n_\infty - \frac{r}{R}(n_\infty - n_r) \tag{3}$$

1a. *Growth of mass by Conensation*

The flux of molecules onto the surface of the droplet is $D(\partial n/\partial R)_{R=r}$, so the rate of mass increase is:

$$\frac{dm}{dt} = 4\pi r^2 D\left(\frac{\partial n}{\partial R}\right)_{R=r} m_0 \tag{4}$$

So we can obtain the change of mass in time as:

$$\frac{dm}{dt} = 4\pi r D(n_\infty - n_r)m_0 \quad (5)$$

$$\frac{dm}{dt} = 4\pi r D(\rho_v - \rho_{vr}) \quad (6)$$

were ρ_v is the ambient vapor density and ρ_{vr} is the vapor density at the droplet's surface.

1b. *Latent Heat release*

Associated with condensation is the release of latent heat which raises the droplet temperature above the ambient value. The difusion of heat away from the droplet is given by

$$\frac{dQ}{dt} = 4\pi r K(T_r - T) \quad (7)$$

where T is the ambient temperature and T_r is the temperature at the surface of the droplet, and K is the coefficient of thermal conductivity of the air.

Condensation of water vapor releases heat ml and the rate of release is $l\frac{dm}{dt}$. The latent heat of condensation is transported by diffusion away from the droplet (assuming no changes in temperature with time -steady state). We can then derive the following equation:

$$\frac{dQ}{dt} = l\frac{dm}{dt} \quad (8)$$

$$\boxed{\frac{\rho_v - \rho_{vr}}{T_r - T} = \frac{K}{Dl}} \quad (9)$$

Here T_r and ρ_{vr} are unknown, but we know the expression for the equilibrium pressure over a droplet:

$$\boxed{\rho_{vr} = \frac{e_s`(r)}{R_v T_r} = \frac{e_s(T_r)}{R_v T_r}\left(1 + \frac{a}{r} - \frac{b}{r^3}\right)} \quad (10)$$

These two equations can be solved simultaneously for T_r and ρ_{vr}, and then solve for the growth. However, there is a simplified expression for the growth (see handout for derivation):

$$\boxed{r\frac{dr}{dt} = \frac{(S-1) - \frac{a}{r} + \frac{b}{r^3}}{F_k + F_d}} \tag{11}$$

where $S = e/e_s$ is te ambient saturation ratio. $F_k = \left(\frac{l}{R_v T} - 1\right)\frac{l\rho_l}{KT}$, is the thermodynamic-heat conduction term. and $F_d = \frac{\rho_l R_v T}{D e_s(T)}$ is the vapor diffusion term. For very large radii:

$$\boxed{r\frac{dr}{dt} = \frac{(S-1)}{F_k + F_d}} \tag{12}$$

Integrating, we get:

$$r^2(t) = r^2(t_0) + \frac{2(S-1)}{F_k + F_d}(t - t_0) \tag{13}$$

This leads to a narrowing of drop size distribution with time. Initially, droplets that form on large condensation nucleus grow faster, but after a certain radius they homogenize. We can also use the equation for evaporation and discriminate between cloud and rain droplets.

2. Collision and Coalescence

Collisions are influenced by gravitational, electrical and aerodynamic forces. Gravitational effects dominate in clouds where large droplets capture small ones.

collision efficiency (E_{coll}) = (# collisions) / (# encounters in geometric sweep)
coalescence efficiency (E_{coal})= (# coalescences) / (# collisions)
collection efficiency (E) = $E_{coll} \times E_{coal}$

Theoretical studies generally assume that the $E = E_{coll}$, for a coalescence efficiency of 1.

2a. *Droplet terminal fall speed*

i. The drag force on a sphere of radius r is:

$$F_R = \frac{\pi}{2} r^2 u^2 \rho C_D \tag{14}$$

u is the velocity of the sphere relative to the fluid. ρ is the fluid density and C_D is the drag coefficient. In terms of the Reynolds number $Re = \frac{2\rho ur}{\mu}$, where μ is the dynamic viscosity. The Reynolds number is a non-dimensional ratio of inertial to viscous forces.

$$F_R = \frac{\pi \mu r u C_D Re}{4} \tag{15}$$

ii. The gravitational force minus buoyancy on a sphere of radius r is:

$$F_A = \frac{4}{3}\pi r^3 g(\rho_l - \rho) \tag{16}$$

iii. Terminal fall speed occurs when $F_A = F_R$

$$\frac{4}{3}\pi r^3 g(\rho_l - \rho) = \frac{\pi \mu r u C_D Re}{4} \tag{17}$$

if $\rho_l >> \rho$

$$u^2 = \frac{8rg\rho_l}{3\rho C_D} \tag{18}$$

$$u = \frac{16r^2 g\rho_l}{3\mu C_D Re} \tag{19}$$

For very small Reynolds numbers $\frac{C_D Re}{24} = 1$

$$u = \frac{2r^2 g\rho_l}{9\mu} \tag{20}$$

For droplets $\approx 30\mu m$, $\boxed{u = k_1 r^2}$. This quadratic dependence of fall speed on size is called Stoke's Law. $k_1 = 1.19 \times 10^6 s^{-1} cm^{-1}$.

For high Reynolds numbers, C_D is independent of Re, and $C_D = 0.45$. $\boxed{u = k_2 r^{1/2}}$ and $k_2 = 2.2 \times 10^3 \left(\frac{\rho_0}{\rho}\right)^{1/2} cm^{1/2} s^{-}1$. Where ρ is the air density and ρ_0 is a reference density of $1.2kg/m^3$. The region in between has a linear relation $\boxed{u = k_3 r}$ where $k_3 = 8 \times 10^3 s^{-1}$.

2b. *Collision Efficiency*

- A drop of radius R overtaking a drop of radius r.
- An object with zero inertia would be swept aside.
- The relative importance of the balance between inertial and aerodynamic forces and the separation x between drop centers.
- At a critical distance x_0 the droplet makes a grazing collision. At $x < x_0$ the droplet collides.

The effective collision cross section of the collector drop of radius R is $[\pi x_0^2]$. The geometric collision cross section $[\pi(R+r)^2]$

$$E_{coll}(R,r) = \frac{x_0^2}{(R+r)^2} \tag{21}$$

if $x_)< (r+R)$ not all drops in the geometrical collision cross section will collide with the collector drop. These calculations are mathematically involved.

- Collision efficiency is small for small r/R
- Increases as r/R increases up to 0.6
- At around 0.6, the radii are similar and relative velocities are small, the flow fields interact strongly
- at r/R close to 1, there is a "wake effect" and the efficiency can be greater than one.

2c. *Growth Equations*

The drop of radius R is falling at terminal speed through a population of smaller droplets. the geometric collision volume swept is calculated as:

$$V_c oll = \pi(R+r)^2[U(R) - u(r)]\Delta t \tag{22}$$

The average number of droplets with radii between r and dr is $n(r)dr$, so the average number of droplets collected in unit time is:

$$N_c oll = \pi(R+r)^2[U(R) - u(r)]E(R,r)n(r)dr \tag{23}$$

for $r = 100\mu m$ assume coalescence efficiency =1, so the collection and collision efficiencies are equal.

The rate of increased volume with time is calculated as the number of droplets times the volume of droplets integrated over all droplet sizes.

$$\frac{dV}{dt} = \int_0^R \frac{4}{3}\pi r^3 \pi (R+r)^2 [U(R) - u(r)] E(R,r) n(r) dr \tag{24}$$

We can express it in terms of the radius of the collector drop:

$$\frac{dR}{dt} = \frac{\pi}{3} \int_0^R \frac{(R+r)^2}{R^2} [U(R) - u(r)] n(r) r^3 E dr \tag{25}$$

If the droplets are much smaller than the collector drop $u(r) \approx 0$ and $(R+r) \approx R$

$$\frac{dR}{dt} = \frac{U(R)\overline{E}M}{4\rho_l} \tag{26}$$

Where $M = V_l \rho_l$ is the cloud liquid water and $V_l = \frac{M}{\rho_l}$, $\overline{E}$ is the effective average of collision efficiency for droplet population.

$U(R)$ increases as R increases, dR/dt increases with increasing R. For this reason, at the beginning condensation dominates, but after a certain threshold, growth by collisions dominates.

2d. *Change of drop size with Altitude*

The relative speed of the droplet determines the distance traveled:

$$\frac{dz}{dt} = U - r(R) \tag{27}$$

Where U is the updraft speed and $u(R)$ is the fall speed of the drop.

$$\frac{dR}{dz} = \frac{\overline{E}Mu(R)}{4\rho_l(U - r(R))} \tag{28}$$

If $U << u(R)$ then $\frac{dR}{dz} = \frac{-\overline{E}M}{4\rho_l}$ These equations describe droplet growth as a continuous collection process. We can also use it to determine the value of H (height) for any radius R_H. If the radius of the collector drop and any height H is R_H and the radius at cloud base is R_0, If we assume M is independent of z (homogeneous cloud liquid water) - then

$$H = \frac{4\rho_l}{M}\left[\int_{R_0}^{R_H} \frac{U}{u(R)\overline{E}}dR - \int_{R_0}^{R_H} \frac{dR}{\overline{E}}\right] \tag{29}$$

- For small droplets $U > u(R)$ and the first integral dominates and H increases as R_H increases so the drop growing by collision is carried upward.
- For $U < u(R)$, H decreases with increasing R_H and the drop begins to fall. Eventually passing through cloud base and reaching the ground if it doesn't evaporate along the way.