The Power Spectrum

The plot of $c_k^2/2$ vs. k is called the power spectrum of $y(t)$

Frequency spectrum if t is time
Wavenumber spectrum if t is space

The 'line spectrum' (i.e. computing the Fourier transform using the entire length of record) is fine if you have infinite record.

For a finite spectrum, the line spectrum has 3 main drawbacks.

1) Integral values of k do not have any special relationship to population being sampled. Simply chosen by the length of data record, which is usually just what is available.

2) Individual spectral lines have very few degrees of freedom (~ 2 dof)

N data points determine : mean, $N/2$ amplitudes, $N/2 - 1$ phases
(Recall dof relates to samples − # of estimated parameters)

3) Most geophysical data are not truly periodic, but only quasi-periodic, so they're better represented by spectral bands rather than individual spectral lines.

Just about any thing in climate that has the word 'oscillation' in it behaves this way!

e.g. ENSO $\rightarrow$ 3-7 years
PDO $\rightarrow$ 20-30 years

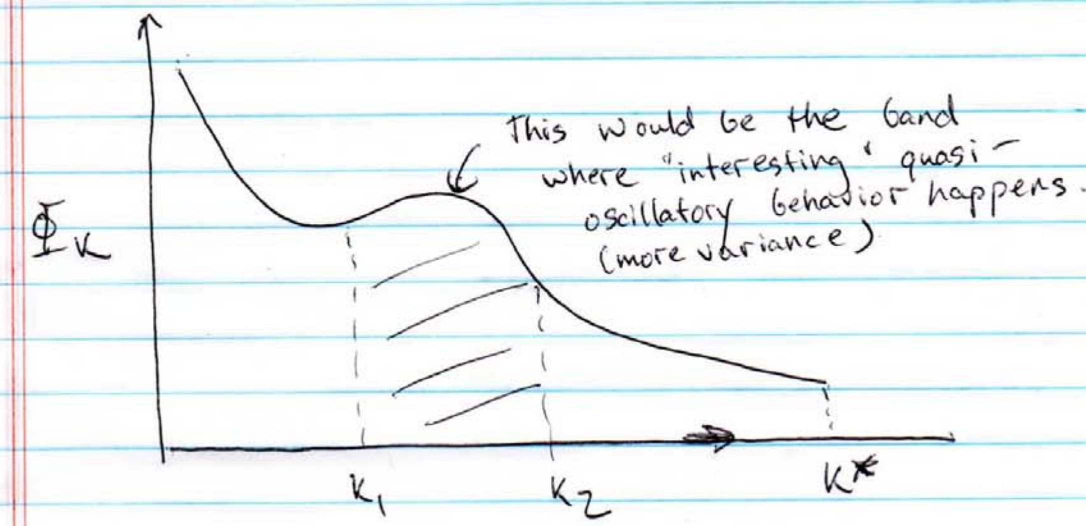
Given these drawbacks, most commonly used is the continuous power spectrum, where variance of y is given per unit frequency!

$$\overline{y'^2} = \int_0^{k^*} \Phi(k)\, dk$$

$\Phi(k)$ = continuous power spectrum
k^* = Nyquist frequency
One cycle per $2\Delta t$
Highest frequency that can be resolved.

Can also write as $\Phi(\omega)$, which is commonly done.

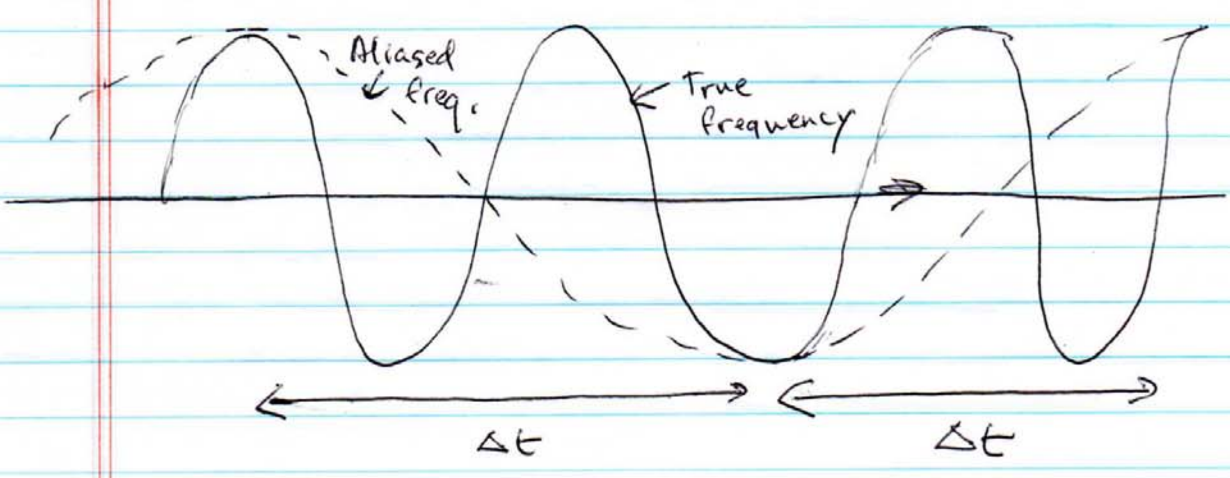
Then for a spectral band:



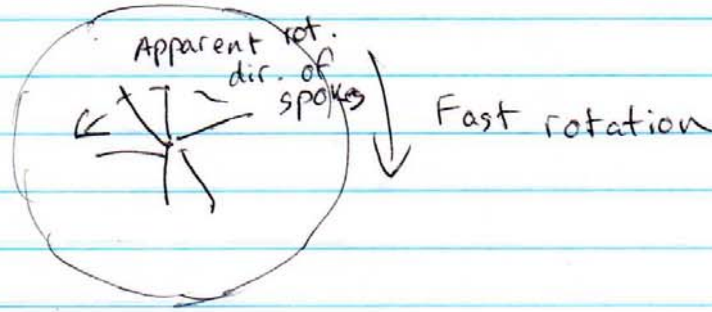
More on the Nyquist frequency & aliasing

Important to know because it is the highest frequency that can be resolved.

If phenomena are occuring at higher frequencies, they will be aliased into lower frequencies



Example of aliasing - a fast rotating wheel, like on a bike or car.



Spokes appear to move against the rotation because the human eye can effectively resolve only so many frames per sec.

Resolution vs. deg. of freedom in estimating $\Phi(k)$

To increase dof in $\Phi(k)$, can do one of 2 things:

1) "Smooth" $\Phi(k)$ by averaging adjacent points in spectrum.
   Tradeoff → lose spectral resolution (larger bandwidth)

2) Average $\Phi(k)$ calculated for subsets of the data:
   Tradeoff → lose information about lowest frequencies in the data.

Always choose method that gives quality results → sig. + reproducible.

Adjusted dof

Degrees of freedom for each spectral estimate is $\sim N/m^*$, where m^* is the number of independent spectral estimates, N = no. of data points

As long as the spectrum is tested against red noise (we'll get to a little later) don't need to decrease N to account for autocorrelation.

Methods of computing power spectra

Method #1: Direct Method

Calculate C_k^2 through harmonic analysis (i.e. as per analytic solutions for the discrete Fourier transform).

Easy to code analytic solutions for C_k.

However, algorithm is not efficient, numerous redundancies in calculation of A_k and B_k.

FFT = Fast Fourier Transform

- Exploits redundancies in the 'direct' method
- Available on software packages (e.g. Matlab) and numerical recipes.

Since the Fourier transform assumes cyclic continuity, typically the ends of the time series are "tapered" by data windowing (more on that later)

Remember, spectral estimates calculated via the direct method have very few degrees of freedom (~2 dof)

Ways to get more dof:

1) Average adjacent estimates of Φ (smoothing)

* [W]hich way to go depends on how much data you have!!

Hartmann ex: 900 day record

450 spectral est. with 2 dof.
Bandwidth $1/900\ \mathrm{day}^{-1}$

Average 10 adjecent estimates
Bandwidth $1/90\ \mathrm{day}^{-1} \rightarrow 20$ dof.

2) Average realizations of Φ (break up the time series)

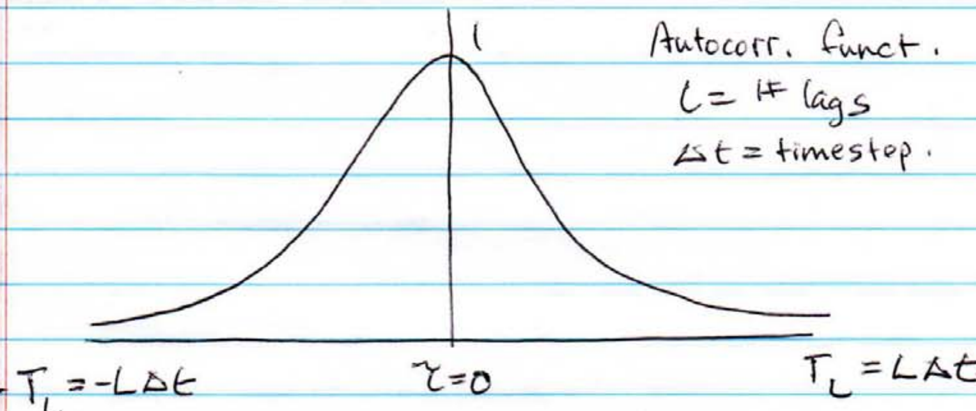
Hartmann ex: 10 series of 900 days.

In this case, if the 10 $\Phi(k)$'s are averaged:
450 spect. est. with 20 dof.
Bandwidth of $1/900\ \mathrm{day}^{-1}$

Method 2: 'Lag Correlation' Method.

As per a theorem by N. Wiener, the autocorrelation function and the power spectrum are Fourier transforms of each other.

Hence, power spectrum can be found by performing harmonic analysis on $r(\tau)$ on the interval $-T_L \le \tau \le T_L$ where T_L is the maximum time lag.



Resolution of the spectrum is given by the choice of L.

Spectrum will have $L+1$ spectral estimates with band width of 1 cycle/$2L\Delta t$

Frequencies resolved at: 0, $\frac{1 \text{ cycle}}{2L\Delta t}$, $\frac{2 \text{ cycles}}{2L\Delta t}$, etc.

The unsmoothed $\Phi(0)$ is the mean of the autocorrelation in the interval $-T_L \rightarrow T_L$ $\Rightarrow$ variance associated with periods too long to fit in the interval

The values of $\Phi(k)$ for $k \geq 1$ are identified with the cosine coefficients obtained from harmonic analysis expansion of $r(\tau)$ (i.e. A_k's)

Since $r(\tau)$ is an even function about $\tau = 0$, the sine coefficients are all equal to zero (all $B_k = 0$).

As before, dof are gained by
- Smoothing spectrum
- Averaging spectral realizations together.

<u>Red noise spectra</u>

It is shown in the Hartmann notes, that in continuous form, get the following Fourier transform pair relating power spectra and lag autocorrelation.

$$\Phi(\omega) = \int_{-T_L}^{T_L} r(\tau)\, e^{-i\omega\tau}\, d\tau$$

$$r(\tau) = \frac{1}{2\pi} \int_{-\omega^*}^{\omega^*} \Phi(\omega)\, e^{i\omega\tau}\, d\omega$$

Can use these relationships to compute power spectrum of red noise:

Autocorrelation function for red noise:

$$r(\tau) = \exp\left(\frac{-\tau}{T}\right) \qquad \begin{array}{l} \tau = \text{lag} \\ T = \text{e-folding timescale.} \end{array}$$

For the power spectrum, then:

$$\Phi(\omega) = \int_{-\infty}^{\infty} \exp\left(\frac{-\tau}{T}\right) \exp^{-i\omega\tau}\, d\tau$$

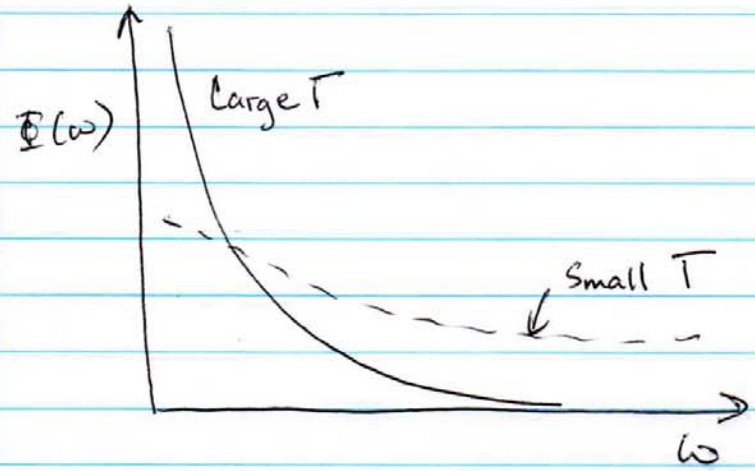
After integration

$$= \frac{-1}{\frac{1}{T} + i\omega} \exp\left\{-\tau\left(\frac{1}{T} + i\omega\right)\right\}_{-\infty}^{\infty}$$

Yields . . .

$$\Phi(\omega) = \frac{2T}{1 + T^2\omega^2}$$

Power spectrum of red noise (graphically)



For white noise, autocorrelation is a delta function of lag

$$r(\tau) = \delta(\tau)$$

$$\Phi(\omega) = \int_{-\infty}^{\infty} \delta(\tau)\, e^{-i\omega\tau}\, d\tau = 1$$

→ Therefore white noise has an equal amount of variance at all frequencies.

Hartmann's notes have idealized plots of different phenomenon and their corresponding spectra.

Plotting the power spectrum

Two basic schemes: linear scale. or log scale.

In the linear scale, plot ω vs. $\Phi(\omega)$ (the power spectral density)

Area under the curve:

$$\int_{\omega_1}^{\omega_2} \Phi(\omega)\, d\omega = \text{Variance explained in } y(t) \text{ between } \omega_1 \text{ and } \omega_2$$

When the frequency interval of interest ranges over several orders of magnitude, it is common to plot $\log(\omega)$ on the abscissa: (natural log)

Linear $\rightarrow \int \Phi(\omega)\, d\omega$

Log $\rightarrow \int \omega\, \Phi(\omega)\, d \ln \omega$

(Hartmann suggests)

In the log plot, $\log(\omega)$ vs $\omega\, \Phi(\omega)$

Effectively stretches the low frequency end and contracts high end.

May just plot against $\Phi(\omega)$, as I've seen that done too ...