Singular Spectrum Analysis (SSA)

* Covered in much greater depth in Mike Evans course... but a good prelude to what we're doing next.

Brief synopsis:

Basic idea: It is like Fourier analysis of a time series – goal is to detect oscillatory behavior.

SSA	Fourier analysis
Basis function T-EOFs (time varying EOFs)	Basis function sines and cosines
Data adaptive	Fixed sinusoidal functions
T-PCs are functions of time.	Fourier amplitudes time-independent

Basic method: Impose an embedding ~~time~~ dimension to a time series

For an imbedding dimension $M = 3$

Fig 11.12

$$x_1, x_2, x_3, x_4, x_5 \ldots x_{n-4}, x_{n-3}, x_{n-2}, x_{n-1}, x_n$$

$$X_{(1)} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad X_{(2)} = \begin{bmatrix} x_2 \\ x_3 \\ x_4 \end{bmatrix} \qquad X_{(n-3)} = \begin{bmatrix} x_{n-3} \\ x_{n-2} \\ x_{n-1} \end{bmatrix} \qquad X_{(n-2)} = \begin{bmatrix} x_{n-2} \\ x_{n-1} \\ x_n \end{bmatrix}$$

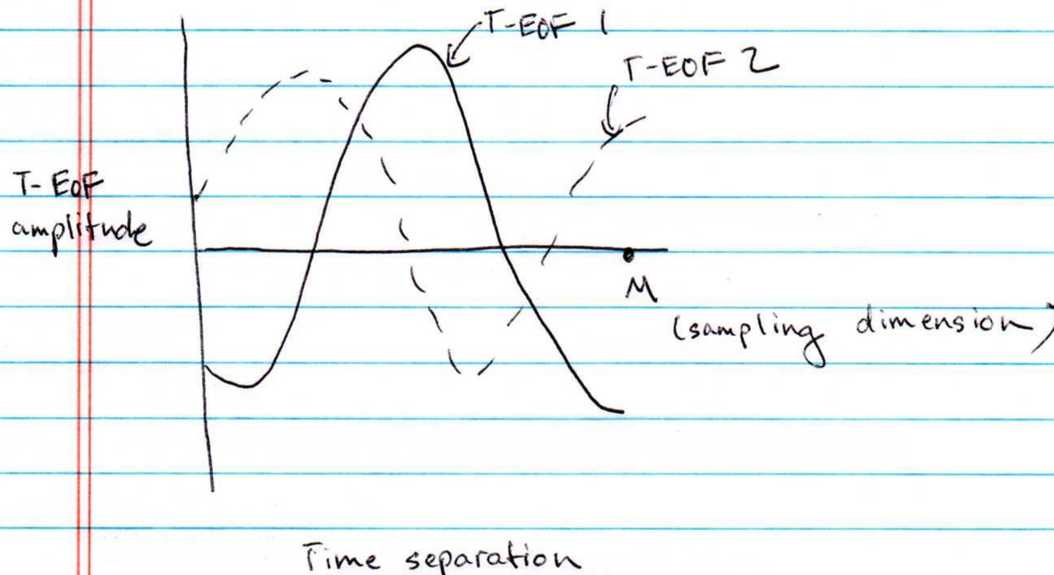
Final matrix is X with $M \times (N-M+1)$ dimensions:

- Then use eigenanalysis (or SVD) to find eigenvectors (T-EOFs) and their time-varying amplitudes (T-PCs)

Eigen vector (T-EOFs): give a basis function (e.g. waveform) of M dimension.

... Once we cover the basics of time series, we'll discuss methods to find significant time-space patterns (e.g. MTM-SVD)

Example in Wilks (Fig 11.14)



In this case, the T-EOFs are not maps, but akin to wave forms in time.

Time series analysis

Objective for this part of course: Find statistically significant frequencies at which data vary (typically in time, but can be in space)

Some definitions:

Stationarity: Implies that statistics of a time series (mean, higher moment statistics) are independent of time

Autocorrelation: Covariance of a time series with itself at another time, as measured by time lag (τ)

Autocovariance function (for time series $x(t)$)

$$\phi(\tau) = \frac{1}{t_2 - t_1 - \tau} \int_{t_1}^{t_2 - \tau} x'(t)\, x'(t+\tau)\, dt$$

t_1 = starting point of time series
t_2 = ending point of time series
"prime" denotes departure from long term mean.

For a discrete time series with elements $k = 1, 2, 3 \ldots N$ with a lag L:

$$\phi(L) = \frac{1}{N-2L} \sum_{K=L}^{N-L} x'_K x'_{K+L} = \overline{x'_K x'_{K+L}}$$

$$L = 0, \underset{\substack{\uparrow \\ \text{lag-1}}}{\pm 1}, \underset{\substack{\uparrow \\ \text{lag-2}}}{\pm 2}, \underset{\substack{\uparrow \\ \text{lag-3}}}{\pm 3}$$

The solution fo $L = 0 \rightarrow \phi(0) = \overline{x'^2} \rightarrow$ variance

The autocorrelation function is the autocovariance normalized by $\phi(0)$. The correlation of a time series with itself at lag L.

Notes on the autocorrelation

1) $-1 \leq r(\tau) \leq 1 \rightarrow$ Varies from -1 to 1, like reg. correlation

2) $r(\tau = 0) = 1$

3) If the time series is <u>not</u> <u>periodic</u>

$$r(\tau) \rightarrow 0 \quad \text{as } \tau \rightarrow \infty$$

4) If the time series is stationary

$$r(+\tau) = r(-\tau)$$

First order autoregression model

Also called 1st order Markov process ("red noise")

A "red noise" time series is defined as:

$$x(t) = a \cdot x(t-\Delta t) + b\,\varepsilon(t)$$

$x \Rightarrow$ standardized variable

$a \Rightarrow$ lies between 0 and 1 and measures the memory from the previous state.

$\varepsilon(t) \Rightarrow$ random variable drawn from standard normal distribution

$\Delta t \Rightarrow$ time between data points

Red noise means "today is like yesterday + some noise" (eg. for weather)

To find a & b

"a" is found by multiplying the above equation by $x(t-\Delta t)$ and taking the time average.

$$\overline{x(t)\,x(t-\Delta t)} = a\,\underbrace{\overline{x(t-\Delta t)\,x(t-\Delta t)}}_{1} + b\,\underbrace{\overline{\varepsilon(t)\,x(t-\Delta t)}}_{0}$$

$$a = \overline{x(t)\; x(t-\Delta t)} = r(\Delta t)$$

$r(\Delta t) = a =$ autocorrelation of x at lag Δt

Since the time series $x(t)$ and $\varepsilon(t)$ have unit variance

$$a^2 + b^2 = 1 \qquad b = \sqrt{1-a^2}$$

The autocorrelation for a 'red noise' time series at lag $\Delta t = 1$ is a

What about for ~~the~~ the next timestep? (in the future) $(2\Delta t)$

$$x(t+\Delta t) = a\, x(t) + b\, \varepsilon(t)$$

Substitute for $x(t)$

$$= a^2\, x(t-\Delta t) + ab\, \varepsilon(t) + b\, \varepsilon(t)$$

$$= a^2\, x(t-\Delta t) + (a+1)\, b\, \varepsilon(t)$$

Multiply by $x(t-\Delta t)$ and take time average, the term with b goes to zero as before

$$\overline{x(t+\Delta t)\, x(t-\Delta t)} = a^2\, \overline{x(t-\Delta t)\, x(t-\Delta t)}$$

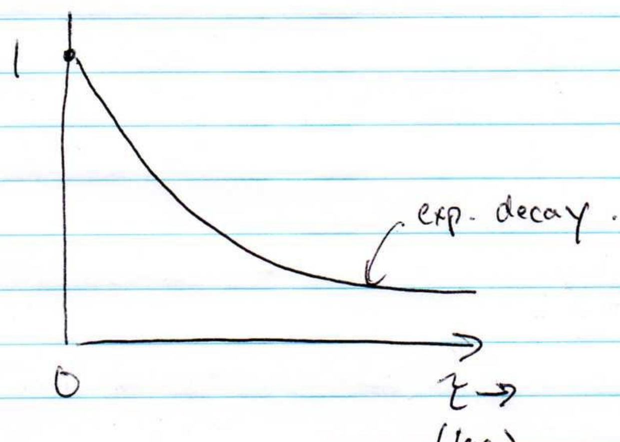
$$\underbrace{\overline{x(t+\Delta t)\, x(t-\Delta t)}}_{\text{lag-2 autocorrelation.}} = a^2 = r^2(\Delta t)$$

Similarly .. Autocorrelation at lag $3\Delta t$ is a^3

" " " lag $4\Delta t$ is a^4

$$r(\tau = n\Delta t) = r^n(\Delta t)$$

Autocorrelation function of red noise decays exponentially for increasing lag $(\tau = n\Delta t)$



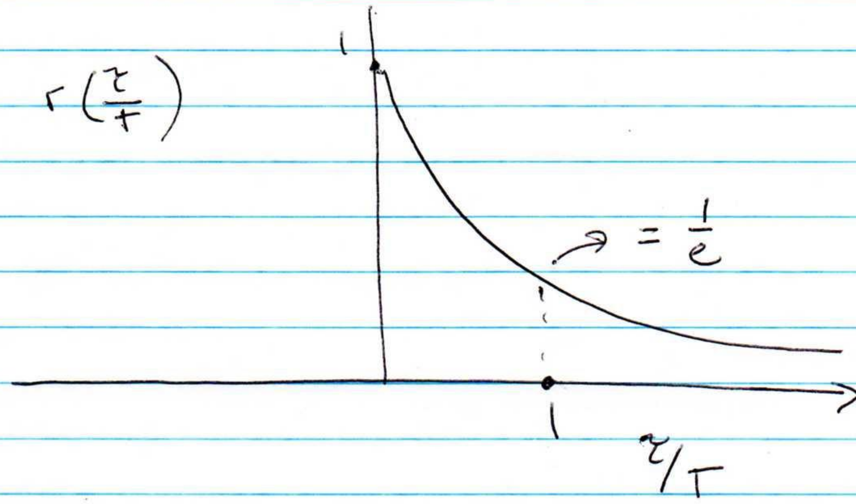
Since the autocorrelation $r(\tau)$ decays exponentially, can define an e-folding timescale, when r drops to $r(\tau=0)\cdot\exp(-1)$

$$\boxed{T = \frac{-\Delta t}{\ln a}} \rightarrow$$ e-folding time scale of autocorrelation function.

if $\Delta t = 1$, lag-1 autocorrelation $= e^{-1} = 0.37$, $T = 1$

" " " $= e^{-0.5} = 0.6$, $T = 2$

Graphically



White noise

Whereas red noise is defined as:

$$x(t) = a\,x(t-\Delta t) + b\,\varepsilon(t)$$

White noise is when $a = 0$
(i.e. no autocorrelation)

Characteristics of white noise:

1) No persistence (zero autocorrelation @ all lags)
2) Equal power at all frequencies