Summary of EOF analysis via eigenanalysis of the covariance matrix

1) If data are in the form $M \times N$ where M = time, N = space, calculate the temporal covariance matrix $(A^T A)$

2) Diagonalize $A^T A$. The 1st EOF is the eigenvector with highest λ, etc.
(need a computer ...)

3) Find PCs by projecting A onto the eigenvectors E.

4) The fraction of variance explained by each EOF/PC pair is:

$$\frac{\lambda_i}{\sum_{i=1}^{N} \lambda_i} = \frac{\text{Eigenvalue of given EOF}}{\text{Sum of all eigenvalues}}.$$

No other combination of k predictors can explain a larger fraction of the variance than the 1st k PCs.

Other notes - Interchangability.

* In case of AA^T, the first eigenvector is the leading PC of A.

$\left.\begin{array}{l} {}^{*}A\,A^{T} \\ A^{T}A \end{array}\right\}$ share the same eigenvalues

EOFs and PCs are completely interchangable with regard to the matrix manipulation.

Using SVD to calculate EOFs / PCs

SVD = Singular Value Decomposition

Why use this method?

- → It is how most EOF/PC routines that are canned computer programs work.
- → Code for it is straightforward, easily accessible. A single command in Matlab!
- → If we understand basically what the SVD routine is doing, it's fine to use it as a "black box" and focus our attention to physical interpretation of the results.

How does it work?

Any rectangular matrix can be represented as the product of 3 special matrices.

$$A\ (M \times N) = U\ (M \times M) \cdot \Sigma\ (M \times N) \cdot V\ (N \times N)$$

"M" = time
"N" = space.

This is denoted in matrix notation

$$\boxed{A = U \Sigma V^T}$$

The decomposition of A into U, Σ, V^T must satify the following relationships:

(1) $AA^T u_i = \sigma_i u_i \Rightarrow C_{M \times M} u_i' = \sigma_i' u_i'$

(2) $A^T A v_i = \sigma_i v_i \Rightarrow C_{N \times N} v_i' = \sigma_i' v_i'$

Where u_i and v_i are the i^{th} columns of matrices U and V, respectively.

Physical interpretation

(1) $\rightarrow$ The eigenvectors (u_i) of the covariance matrix based on the time dimension (M)

Columns in U matrix (u_i)
= Eigenvectors of AA^T = PCs of A.

(2) $\rightarrow$ The eigenvectors (v_i) of the covariance matrix based on the space dimension (N).

Columns in V matrix (v_i)
= Eigenvectors of $A^T A$ = EOFs of A.

(3) $\rightarrow$ The σ_i values correspond to the square roots of the eigenvalues (λ_i)
(WILL PROVE THIS LATER...)

In the case $M = N$ and A is full rank,

$$\Sigma = \begin{array}{cc} \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ & & \ddots \\ 0 & 0 & \sigma_N \end{bmatrix} & M \\ N & \end{array}$$

If $M > N$

$$\Sigma = \begin{array}{ccc} \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ & & \ddots \\ 0 & 0 & \sigma_N \\ 0 & 0 & 0 \end{bmatrix} & \begin{matrix} \uparrow \\ N \\ \downarrow \\ N+1 \\ \downarrow \\ M \end{matrix} & M \\ N & & \end{array}$$

Null space.

<u>Notes on "i"</u>

The largest possible rank of A is $q = \min(M, N)$. All columns in U (or V) where $i \geq q$ lie in the nullspace of A.

Additional notes:

- Columns in U and V matrices (u_i and v_i) must be orthogonal
- Columns in U and V span R^m and R^N, respectively
- Columns in U, V are orthonormal.

What about Σ matrix?

Σ matrix can be viewed as comprising 2 matrices, depending on the relative size of $M \times N$.

For $M > N$
$$\Sigma = \begin{array}{c} \leftarrow N \rightarrow \\ \begin{bmatrix} D \\ 0 \end{bmatrix} \end{array} \begin{array}{c} \uparrow \\ M \\ \downarrow \end{array}$$

$D = N \times N$ diagonal matrix

$0 = (M-N) \times N$ matrix of zeros.

The first q elements (where q = rank of A) of D are the singular values of A. Singular values are the square roots of the eigenvalues of A. ($\sigma_i^2 = \lambda_i$)

The $q > N$ diagonal elements are zeros, corresponding to the nullspace of A.