Properties of eigenvectors

- The eigenvectors of a matrix form a linearly independent set

- If A $(M \times M)$ has full rank, then the eigen vectors form the basis vectors for $\mathbb{R}^m$.

- If A is less than full rank with rank $q < M$, the first q eigenvectors span the subspace of A, the remaining span the nullspace of A.

- If A $(M \times M)$ is real, symmetric, the eigenvectors not only form a linearly independent set, but they are orthogonal
  → Important!!

Proof: Let C be a real, symmetric matrix. For the eigenvectors e_k and e_j:

(Hartmann)

$$C e_j = \lambda_j e_j$$

$$C e_k = \lambda_k e_k$$

Transpose top equation and multiply by e_k.

$$e_j^T C^T e_k = \lambda_j e_j^T e_k$$

Multiply bottom equation by e_j^T

$$e_j^T C e_k = \lambda_k e_j^T e_k$$

Subtract the two equations:

$$e_j^T C^T e_k - e_j^T C e_k = (\lambda_j - \lambda_k) e_j^T e_k.$$

If $C = C^T$ (symmetric matrix)

$$0 = (\lambda_j - \lambda_k) e_j^T e_k.$$

Therefore unless the eigenvalues are equal (i.e $\lambda_j = \lambda_k$), then

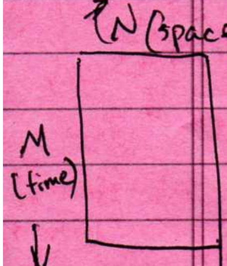
$$e_j^T e_k = 0$$

Therefore the eigenvectors must be orthogonal if the eigenvalues are distinct.

Introduction to EOF's

EOFs = Empirical Orthogonal Functions

Basic premise: Say have a data matrix with M observations of length; N state vectors (M = time, N = space)



We want to find the $1 \times N$ state vector (or the spatial pattern) that explains the largest fraction of the total variance of the data matrix.

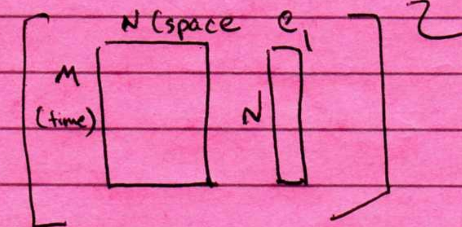
Let e_1 (the 1st EOF) denote the vector (or spatial pattern) that has the highest possible resemblance to the whole ensemble of data $M \times N$. (and so on for EOF 2, 3, etc.)

Notes

- "Resemblance" is given by the mean projection of e_1 onto the ensemble of data.
- The projection is squared. (similar to squaring the error in linear regression)
- The vector e_1 is normalized such that it's magnitude = 1. So only the direction of e_1 will impact the projection.
- ~~EOFs~~ Dominant EOFs are sometimes (loosely) referred to as the dominant (spatial) modes.

Find e_1 that has the highest possible resemblance to the ensemble of state vectors

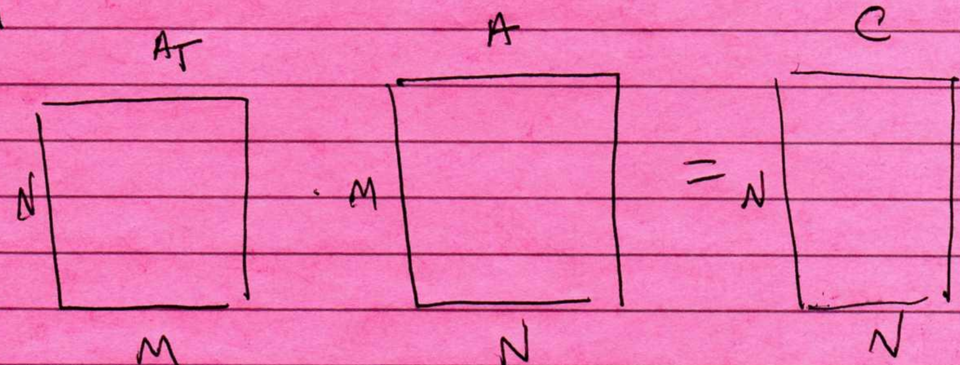
Want to maximize:

$\lambda = \frac{1}{N}$ 

In matrix notation

$$\lambda = \frac{1}{N}\left(Ae_1\right)^2 = \frac{1}{N} e_1^T \underbrace{A^T A}_{C \,=\, \text{Covariance matrix} \cdot N} e_1$$

Recall



$$\lambda = e_1^T C e_1$$

The eigenvectors have the following orthonormality property:

$$e^T e = I$$

Therefore can express this as an eigenvalue problem in the form:

$$Ce_1 = \lambda e_1 \quad \text{or} \quad (C - \lambda I) e_1 = 0.$$

What does this tell us about e_1 and λ?

1) e_1 must be an eigenvector of the covariance matrix ($C = A^T A$) with the corresponding eigenvalue λ.

2) e_1 is the eigenvector that corresponds to the largest λ.

3) The eigenvectors of the covariance matrix C are called empirical orthogonal functions (EOFs)

4) The pattern that explains the largest fraction of the variance in a dataset A ($M \times N$) is the eigenvector corresponding to the largest eigenvalue of the covariance matrix ($A^T A$).

The subsequent EOFs (2^{nd}, 3^{rd}, etc.) are those corresponding to the second, third, etc. largest λ of $C = A^T A$.

$$C\underline{E} = \underline{E}\,L$$

C = symmetric square matrix (cov. matrix)
$\underline{E}$ = matrix with columns corresponding to eigenvectors of covariance matrix.
L = diagonal matrix with the eigenvalues λ along the diagonal.

$$\begin{array}{cc} & N \text{ (space)} \\ \begin{array}{c} N \\ \text{(space)} \end{array} & \begin{bmatrix} \overline{x_1^2} & \overline{x_1 x_2} & \cdots \\ \overline{x_2 x_1} & \overline{x_2^2} & \cdots \\ & & \ddots \end{bmatrix} \end{array} \rightarrow \text{Cov. matrix } (C)$$

(Diagonal elements = variances
~~off-diagonal elements =~~
covariances)

$$\begin{array}{cc} & \begin{array}{ccc} \text{1st EOF} & \text{2nd EOF} & \end{array} \\ N & \begin{bmatrix} e_1 & e_2 & \cdots & e_N \\ \downarrow & \downarrow & & \\ & & & \end{bmatrix} \end{array} \rightarrow \text{Eigenvectors } (\underline{E})$$

Each of length (N)

$$N \quad \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \rightarrow \text{Eigenvalues in diagonal matrix } L.$$

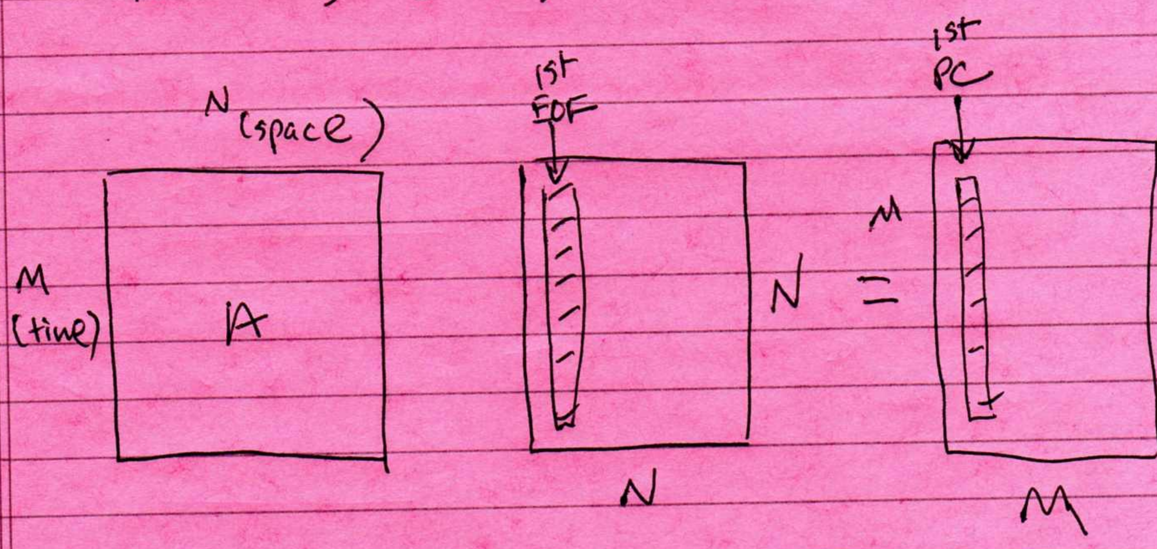
Eigenanalysis acts to transform the covariance matrix into a coordinate space where the "new" matrix is reorganized along the diagonal.

→ Eigen analysis diagonalizes the covariance matrix.

<u>Time</u> <u>series</u> of <u>EOFs</u>. – Principal components (PC)

Time series of EOFs, or principal components, found by projecting the data onto the eigen vectors.

$$A\,(M \times N) \cdot e \Rightarrow \text{PC time series.}$$



The projection of A·E yields a coefficient matrix of the expansion of the data in terms of the eigenvectors. The exp. coef. time series are referred to as the PC time series.