Contents

1. The Richardson Number

1a. *Flux Richardson Number*

The ratio of the buoyant production term (Term III) and the mechanical production term (Term IV) is called the **Flux Richardson Number** (R_f). This number characterizes the thermal stability of the flow.

$$R_f = \frac{\frac{g}{\theta_v}\overline{(w'\theta_v')}}{\overline{(u_i'u_j')}\frac{\partial \overline{U_i}}{\partial x_j}} \tag{1}$$

The denominator consists of 9 terms. We assume horizontal homogeneity and neglect subsidence:

$$R_f = \frac{\frac{g}{\theta_v}\overline{(w'\theta_v')}}{\overline{(u'w')}\frac{\partial \overline{U}}{\partial z} + \overline{(v'w')}\frac{\partial \overline{V}}{\partial z}} \tag{2}$$

Remember, the denominator is usually negative.

- $R_f > 0$ for statically stable flows
- $R_f < 0$ for statically unstable flows
- $R_f = 0$ for statically neutral flows

At the critical value of $R_f = +1$, the mechanical production rate balances the buoyant consumption.

- $R_f < +10$static stability is insufficient to prevent the mechanical generation of turbulence, flow is dynamically unstable. (Statically unstable flow is always dynamically unstable).
- $R_f > +1$ flow becomes laminar (dynamically stable)
- $R_f = 0$ for statically neutral flows

1b. *Gradient Richardson Number*

The value of the turbulent correlations could be expressed as being proportional to the lapse rate, and the turbulent momentum flux can be proportional to the wind gradient: $\overline{w'\theta_v'} \propto \frac{\partial \theta_v}{\partial z}$, $\overline{w'u'} \propto \frac{\partial \overline{U}}{\partial z}$ and $\overline{w'v'} \propto \frac{\partial \overline{V}}{\partial z}$. This is the basis of K-theory, that we will discuss later. When substituting into equation 4, we get the Gradient Richardson Number:

$$R_i = \frac{\frac{g}{\theta_v}\frac{\partial \overline{\theta_v}}{\partial z}}{\left(\frac{\partial \overline{U}}{\partial z}\right)^2 + \left(\frac{\partial \overline{V}}{\partial z}\right)^2} \tag{3}$$

Laminar flow becomes turbulent when $R_i < R_c \approx .21$, and turbulent flow becomes laminar when $R_i > R_T \approx 1$. There is a hysteresis effect.

1c. *Bulk Richardson Number*

When measuring wind shear and temperature gradients, meteorologists approximate the gradients by measurements at discrete heights:

$$R_B = \frac{\frac{g}{\overline{\theta_v}}\frac{\Delta\overline{\theta_v}}{\Delta z}}{\left(\frac{\Delta\overline{U}}{\Delta z}\right)^2 + \left(\frac{\Delta\overline{V}}{\Delta z}\right)^2} = \frac{g\Delta\overline{\theta}_v\Delta z}{\overline{\theta_v}((\Delta\overline{U})^2 + (\Delta\overline{V})^2} \tag{4}$$

This is the form most frequently used. The values of the critical Richardson number don't apply to these finite differences across thick layers. The thinner the layer, the closer the value to the theory.

2. The Obukhov Length

This is a very important parameter in the surface layer. Remember that in the surface layer we can assume constant flux with height (by definition). Let's multiply the TKE equation by $(-kz/u_*^3)$ where k is the *von Karman constant* ≈ 0.4. We assume that all the turbulent fluxes are equal to their values at the surface, and focus only on the buoyant production and mechanical production terms.

$$... = \underbrace{\frac{-kzg}{u_*^3}\frac{\overline{w'\theta_v'}}{\overline{\theta_v}}}_{III} + \underbrace{\frac{kz(\overline{u_i'u_j'})_s}{u_*^3}\frac{\partial\overline{U_i}}{\partial x_j}}_{IV} + ... \tag{5}$$

Term III is assigned the symbol $\zeta \equiv \frac{z}{L}$ where L is the ***Obukhov length***.

$$\zeta = \frac{z}{L} = \frac{-kzg}{u_*^3}\frac{\overline{w'\theta_v'}}{\overline{\theta_v}} \tag{6}$$

The Obukhov Length is given by

$$L = \frac{-\overline{\theta_v}u_*^3}{kg(\overline{w'\theta_v'})_s} \tag{7}$$

Physical interpretation: proportional to the height above the surface at which buoyant factors first dominate over mechanical (shear) production of turbulence. Buoyant and shear production terms are approximately equal at $z = -0.5L$. ζ is a surface layer scaling parameter.

- $\zeta < 0$ Unstable

- $\zeta > 0$ Stable

3. Dimensionless Gradients

Let's look now at Term IV of the normalized TKE, with a system aligned with the mean wind, assuming horizontal homogeneity and neglecting subsidence. We use the definition of $u*^2 = -(\overline{u'w'})_s$.

$$... = ... - \underbrace{\frac{kz}{u_*}\frac{\partial \overline{U_i}}{\partial z}}_{IV} + ... \tag{8}$$

Based on this term, we define a dimensionless wind shear:

$$\phi_m = \frac{kz}{u_*}\frac{\partial \overline{U_i}}{\partial z} \tag{9}$$

This is used for surface-layer wind profiles and momentum fluxes. We use ϕ_m in similarity theory. By analogy, the dimensionless lapse rate ϕ_H and dimensionless humidity gradient ϕ_H:

$$\phi_H = \frac{kz}{\theta_*^{SL}}\frac{\partial \overline{\theta}}{\partial z} \tag{10}$$

$$\phi_E = \frac{kz}{q_*^{SL}}\frac{\partial \overline{q}}{\partial z} \tag{11}$$

All of these non-dimensional numbers are= 1 for *Neutral Conditions*.

4. Surface Layer Scaling

Recall when we spoke about K-Theory when discussing First Order Closure:

We can define a ***mixing length***, l by $l^2 = c\overline{z'^2}$.

In the surface layer eddies are limited by the earth's surface. It is assumed that $l^2 = k^2 z^2$ where k is the von Karman constant, so:

$$K_E = K_H = K_m = l^2 \left|\frac{\partial \overline{U}}{\partial z}\right| = k^2 z^2 \left|\frac{\partial \overline{U}}{\partial z}\right| \tag{12}$$

In reality, the origin of z for a rough surface because the protrusion of roughness elements above the substrate surface displaces the entire flow upwards. We define the displaced height $z = Z - d$, where d is the zero-displacement height and Z is the height above the substrate surface (height above the actual ground surface).

Hence, for *neutral conditions* with *no buoyancy*, in the *surface layer* (assuming that the stress remains constant throughout the surface layer) we recall the friction velocity, choosing the x-axis appropriately, reduces to:

$$u_*^2 = -\overline{u'w'} = K_m \frac{\partial \overline{U}}{\partial z} = k^2 z^2 \left(\frac{\partial \overline{U}}{\partial z} \right)^2 \tag{13}$$

$$u_* = kz \left(\frac{\partial \overline{U}}{\partial z} \right) \tag{14}$$

Integration gives the famous log-wind profile for neutral conditions:

$$\frac{k\overline{U}}{u_*} = ln(z) + cnt \tag{15}$$

5. Monin-Obukhov Similarity Theory

We can take into account the influence of buoyancy through the Richardson number R_f or the Obukhov Length L. The way this is generally done is by taking the dimensionless gradients we had expressed before, which are equal to 1 in neutral conditions, and expressing them as functions of ζ for non-neutral conditions:

Based on this term, we define a dimensionless wind shear:

$$\phi_m(\zeta) = \frac{kz}{u_*} \frac{\partial \overline{U_i}}{\partial z} \tag{16}$$

$$\phi_H(\zeta) = \frac{kz}{\theta_*^{SL}} \frac{\partial \overline{\theta}}{\partial z} \tag{17}$$

$$\phi_E(\zeta) = \frac{kz}{q_*^{SL}} \frac{\partial \overline{q}}{\partial z} \tag{18}$$

(NOTE: substitute the variables for their values in the neutral BL and verify

that these variables=1 in the neutral BL)
The forms of the ϕ functions have been extensively studied using observations from many experiments. Observations suggest that:

For $-5 < \zeta < 0$

$$\begin{aligned} \phi_m(\zeta) &= (1-16\zeta)^{-1/4} && (19) \\ \phi_H(\zeta) &= \phi_E(\zeta) = (1-16\zeta)^{-1/2} && (20) \end{aligned}$$

For $0 < \zeta < 1$

$$\phi_m = \phi_H = \phi_E = 1 + 5\zeta \qquad (21)$$

5a. *Integral forms of the flux-gradient relations*

i. Wind For the general, non-neutral case, the surface layer wind profile can be obtained by integrating equation 16:

$$\begin{aligned} \frac{\partial \overline{U}}{\partial z} &= \frac{u_* \phi_m}{kz} && (22) \\ \overline{U}(z) &= \frac{u_*}{k} \int_{z_0}^{z} \left(\frac{dz'}{z'} - \frac{dz'}{z'} + \phi_m \frac{dz'}{z'} \right) \\ &= \frac{u_*}{k} \left[ln\frac{z}{z_0} - \int_{z_0}^{z} (1-\phi_m) \frac{dz'}{z'} \right] \\ &= \frac{u_*}{k} \left[ln\frac{z}{z_0} - \psi_m(\zeta) \right] \end{aligned}$$

for $0 < \zeta <= 1$ Stable

$$\psi_m = -5\zeta \qquad (23)$$

for $1 < \zeta$ Stable

$$\psi_m = -5ln(z/z_0) \qquad (24)$$

for $\zeta < 0$ ($x = \phi_m^{-1} = (1-16\zeta)^{1/4}$) Unstable

$$\psi_m = 2ln\frac{1+x}{2} + ln\frac{1+x^2}{2} - 2tan^{-1}x + \frac{\pi}{2} \qquad (25)$$

In this form, the effects of buoyancy can be interpreted as a deviation of the wind speed from the neutral value.

- In unstable conditions $0 < \phi < 1$ and $\psi > 0$
- In stable conditions $\phi > 0$ and $\psi < 0$

In the general case where we have two wind measurements at heights 1 and 2, we can extend the above expression to:

$$\overline{U_2} - \overline{U_1} = \frac{u_*}{k}\left[ln\frac{z_2}{z_1} - \psi_m(\zeta_2) + \psi_m(\zeta_1)\right] \tag{26}$$

ii. Temperature In analogous form:

$$\frac{k(\overline{\theta} - \theta_0)}{\theta_*^{SL}} = ln\frac{z}{z_T} - \psi_H(\zeta) \tag{27}$$

$$\frac{k(\overline{\theta_v} - \theta_{v0})}{\theta_{v*}^{SL}} = ln\frac{z}{z_T} - \psi_H(\zeta) \tag{28}$$

Here z_T is the surface scaling length for temperature. Formally $\theta = \theta_0$ at $z = z_T$, and z_T is not necessarily equal to z_0. Notice that we are assuming the same nondimensional numbers apply to potential and virtual potential temperature.

for $0 < \zeta <= 1$ Stable

$$\psi_m = -5\zeta \tag{29}$$

for $1 < \zeta$ Stable

$$\psi_m = -5ln(z/z_0) \tag{30}$$

for $\zeta < 0$ ($y = \phi_H^{-1} = (1 - 16\zeta)^{1/2}$) Unstable

$$\psi_H = 2ln\frac{1+y}{2} \tag{31}$$

In the general case where we have two temperature measurements at heights 1 and 2, we can extend the above expression to:

$$\frac{k(\overline{\theta_2} - \overline{\theta_1})}{\theta_*^{SL}} = ln\frac{z_2}{z_1} - \psi_H(\zeta_2) + \psi_H(\zeta_1) \tag{32}$$

$$\frac{k(\overline{\theta_{v2}} - \overline{\theta_{v1}})}{\theta_*^{SL}} = ln\frac{z_2}{z_1} - \psi_H(\zeta_2) + \psi_H(\zeta_1) \tag{33}$$

iii. Humidity In analogous form:

$$\frac{k(\overline{q} - q_0)}{q_*^{SL}} = ln\frac{z}{z_q} - \psi_E(\zeta) \tag{34}$$

for $0 < \zeta <= 1$ Stable

$$\psi_m = -5\zeta \tag{35}$$

for $1 < \zeta$ Stable

$$\psi_m = -5ln(z/z_0) \tag{36}$$

for $\zeta < 0$ ($y = \phi_H^{-1} = (1 - 16\zeta)^{1/2}$) Unstable

$$\psi_E = 2ln\frac{1 + y}{2} \tag{37}$$

In the general case where we have two humidity measurements at heights 1 and 2, we can extend the above expression to:

$$\frac{k(\overline{q_2} - \overline{q_1})}{q_*^{SL}} = ln\frac{z_2}{z_1} - \psi_E(\zeta_2) + \psi_E(\zeta_1) \tag{38}$$

Observations and theory suggest that $\Phi_E = \Phi_H$ and $\psi_E = \psi_H$ and $z_q = z_T$

5b. *Calculating Fluxes using the Flux Profile Method*

As we have shown before, if the stability and the flux or stress is known in advance, then the flux profile method can be used to solve directly for the wind speed or the potential temperature at any height. However, often these relationships are used in reverse, to estimate the flux knowing the mean wind or temperature profile. This is much more difficult. Fore example, u_* appears in a number of places, explicitly and hidden in L, and L is a function of heat flux, which must be estimated from the temperature profile. Solving these equations involves an iterative approach.

Notice how we can use the above expressions to calculate the fluxes as:

$$u_* = \frac{k[\overline{U_2} - \overline{U_1}]}{\left[ln\frac{z_2}{z_1} - \psi_m(\zeta_2) + \psi_m(\zeta_1)\right]} \tag{39}$$

if z_1 = z_0 then U_1=0 and $\psi_m(\zeta_1) = 0$

$$\overline{(w'\theta')_s} = \frac{-u_* k(\overline{\theta_2} - \overline{\theta_1})}{ln\frac{z_2}{z_1} - \psi_H(\zeta_2) + \psi_H(\zeta_1)} \tag{40}$$

$$\overline{(w'\theta'_v)_s} = \frac{-u_* k(\overline{\theta_{v2}} - \overline{\theta_{v1}})}{ln\frac{z_2}{z_1} - \psi_H(\zeta_2) + \psi_H(\zeta_1)} \tag{41}$$

$$\overline{(w'q')_s} = \frac{-u_* k(\overline{q_2} - \overline{q_1})}{ln\frac{z_2}{z_1} - \psi_E(\zeta_2) + \psi_E(\zeta_1)} \tag{42}$$

Given the mean wind (U), pressure (P), humidity (q) and temperature (T) at a level z and at the ”surface”.

1. From the information given calculate the density of air ρ the latent heat L_v, θ and θ_v
2. Calculate u_*, assuming neutral conditions.
3. Calculate $\overline{w'q'}$
4. Calculate $\overline{w'\theta'}$, $\overline{w'\theta'_v}$
5. Calculate L
6. Begin iteration i
   (a) If $L^i > 0$ conditions are stable - calculate $\psi_E = \psi_H$ and ψ_m
   (b) If $L^i < 0$ conditions are unstable - calculate $\psi_E = \psi_H$ and ψ_m
   (c) If $L^i = 0$ conditions are neutral $\psi_E = \psi_H = \psi_m = 0$
   (d) Re-calculate u_*, $\overline{w'q'}$, $\overline{w'\theta'}$ and L using the relationships that depend on stability ζ
   (e) Calculate the difference in the fluxes $\overline{w'q'}$, $\overline{w'\theta'}$ between this iteration and the previous iteration. If the difference is large, continue to iterate until your answers converge.

6. Bulk Transfer Relations

For practical applications, we use drag and bulk transfer coefficients to relate fluxes to mean properties of the flow.

6a. *Drag Coefficient*

Using the relationship 23, and the definition of friction velocity u_*, a drag coefficient $C_D$is defined as:

$$C_D = \frac{(\overline{u'w'}^2)^{1/2}}{\overline{U}^2} = \frac{u_*^2}{\overline{U}^2} = \frac{k^2}{\left[ln\frac{z}{z_0} - \psi_M(\zeta)\right]^2} \tag{43}$$

$$C_{DN} = \frac{k^2}{\left[ln\frac{z}{z_0}\right]^2} \tag{44}$$

$$\frac{C_D}{C_{DN}} = \left[1 - \frac{\psi_M(\zeta)}{ln\frac{z}{z_0}}\right]^{-2} \tag{45}$$

6b. *Heat Transfer Coefficient*

By analogy with the drag coefficient, a heat transfer coefficient C_H can be defined using the relationship 42 and the definition of $\theta_*^{SL} = -\overline{w'\theta'_s}/u_*$

$$\begin{aligned} C_H = \frac{Q_H}{\overline{U}(\theta_0 - \overline{\theta})} &= \frac{(\overline{\theta'w'}_s)}{\overline{U}(\theta_0 - \overline{\theta})} && (46) \\ &= \frac{(\overline{\theta'w'}_s)}{\frac{-u_*}{k}\left[ln\frac{z}{z_0} - \psi_M(\zeta)\right]\frac{\theta_*^{SL}}{k}\left[ln\frac{z}{z_T} - \psi_H(\zeta)\right]} \\ &= \frac{k^2}{\left[ln\frac{z}{z_0} - \psi_M(\zeta)\right]\left[ln\frac{z}{z_T} - \psi_H(\zeta)\right]} \\ & && (47) \end{aligned}$$

Where Q_H is the kinematic sensible heat which is the sensible heat divided by ρC_p

$$C_{HN} = \frac{k^2}{\left[ln\frac{z}{z_0}\right]\left[ln\frac{z}{z_T}\right]} \tag{48}$$

$$\frac{C_H}{C_{HN}} = \left[\frac{1}{1 - \frac{\psi_M(\zeta)}{ln\frac{z}{z_0}}}\right]\left[\frac{1}{1 - \frac{\psi_H(\zeta)}{ln\frac{z}{z_T}}}\right] \tag{49}$$

Values of C_{DN}/C_{HN} greater than one indicate the more efficient transfer of momentum than heat as the surface is rougher.

6c. *Moisture Transfer Coefficient*

By analogy with the heat transfer coefficient, a heat transfer coefficient C_E can be defined using the relationship 34 and the definition of $q_*^{SL} = -\overline{w'q'_s}/u_*$

$$C_E = \frac{R}{\overline{U}(q_0 - \overline{q})} = \frac{(\overline{q'w'}_s)}{\overline{U}(q_0 - \overline{q})} \tag{50}$$

$$= \frac{(\overline{q'w'}_s)}{\frac{-u_*}{k}\left[ln\frac{z}{z_0} - \psi_M(\zeta)\right]\frac{q_*^{SL}}{k}\left[ln\frac{z}{z_q} - \psi_E(\zeta)\right]}$$

$$= \frac{k^2}{\left[ln\frac{z}{z_0} - \psi_M(\zeta)\right]\left[ln\frac{z}{z_q} - \psi_E(\zeta)\right]} \tag{51}$$

Where R is the kinematic vertical eddy moisture flux.

$$C_{EN} = \frac{k^2}{\left[ln\frac{z}{z_0}\right]\left[ln\frac{z}{z_q}\right]} \tag{52}$$

$$\frac{C_E}{C_{EN}} = \left[\frac{1}{1 - \frac{\psi_M(\zeta)}{ln\frac{z}{z_0}}}\right]\left[\frac{1}{1 - \frac{\psi_E(\zeta)}{ln\frac{z}{z_q}}}\right] \tag{53}$$

7. Aerodynamic Resistances

The drag, heat and mass transfer coefficients discussed above take into account both turbulent transfer and molecular transfer of a property between the surface and a reference height z in the surface layer. For some applications, it is more convenient to replace the transfer coefficients by ”quasi-resistance” parameters.

In this approach, the linking of molecular transfer in the interfacial layer and turbulent transfer in the surface layer is simplified. This relates to the additive property of resistances in series.
By analogy to Ohm's law (resistance = potential difference / current). For any concentration difference $(\gamma_0 - \gamma)$ and flux F_s

$$r_a = (\gamma_0 - \gamma)/F_s \tag{54}$$

r_a has dimensions of sm^{-1}. The reciprocal r_a^{-1} is the conductance.

7a. *Momentum*

From the definition of C_D (equation 43, we define the *bulk aerodynamic resistance* to the transfer of momentum from a level z to the surface $z = z_0$ as:

$$r_{aM} = \frac{\rho(u(z) - u(z_0))}{\tau_s} = \frac{u(z)}{u_*^2} = (C_D u(z))^{-1} \tag{55}$$

As C_D increases or $u(z)$ increases, the resistance increases.

7b. *Heat*

From the definition of C_H , we define the *bulk aerodynamic resistance* to the transfer of heat from the surface $z = z_0$ to a level z as:

$$r_{aH} = \frac{(\theta_0 - \theta)}{H_0} = (C_H u(z))^{-1} \tag{56}$$

7c. *Moisture*

From the definition of C_E , we define the *bulk aerodynamic resistance* to the transfer of moisture from the surface $z = z_0$ to a level z as:

$$r_{aE} = \frac{(q_0 - \theta)}{E_0} = (C_E u(z))^{-1} \tag{57}$$

It is important to note that under near-neutral conditions, the resistance for moisture and heat is higher than for momentum.

Figure 1: Figure 3.8 Garrat

The surface values θ_0 and q_0 must be estimated to use the expressions for the bulk aerodynamic resistance to sensible and latent heat exchange:

$$r_{aH} = \frac{\rho c_p (\theta_0 - \theta)}{H_0} \tag{58}$$

$$r_{aV} = \frac{\rho (q_0 - q)}{E_0} \tag{59}$$