Contents

1. Turbulence Closure Techniques

There are a large number of unknowns in the equations we developed.

1a. *The Closure Problem*

- The number of unknowns is larger than the number of equations
- When we start deriving equations for unknowns, more variables appear.

Closure Problem: " Total statistical description of turbulence requires an infinite set of equations".

Table 1: default

Progn. Eq. for:	Moment	Equation	No. Eqs.	No. Unknowns
$\overline{U_i}$	First	$\frac{\partial \overline{U_i}}{\partial t} = ... - \frac{\partial \overline{u'_i u'_j}}{\partial x_j}$	3	6
$\overline{u'_i u'_j}$	First	$\frac{\partial \overline{u'_i u'_j}}{\partial t} = ... - \frac{\partial \overline{u'_k u'_i u'_j}}{\partial x_k}$	6	10
$\overline{u'_i u'_j u'_k}$	First	$\frac{\partial \overline{u'_i u'_j u'_k}}{\partial t} = ... - \frac{\partial \overline{u'_k u'_i u'_j u'_m}}{\partial x_m}$	10	15

So, for example, for the first order you have 6 unknowns: $\overline{u'^2}$, $\overline{v'^2}$, $\overline{w'^2}$, $\overline{u'v'}$, $\overline{u'w'}$, $\overline{v'w'}$. We can derive prognostic equations for these correlation terms, but then the new equations contain additional triple correlation terms. If we were to derive prognosic equations for these they would contain fourth-order moments...

Solution: use only a finite number of equations, then approximate the remaining unknowns. We name these ***closure approximations*** according to the highest order prognostic equations retained.

If we use the prognostic equation for the mean variables, and approximate the second moments: ***first-order closure***. However, some assumptions use only some of the equations, for example, equations for TKE, temperature and moisture variances along with first order moment equations: ***one-and-a-half order closure***.

There are two major schools of thought of turbulence closure:

local closure the unknown quantity is parameterized by values and/or gradients of known quantities at the *same* point. - Analogous to molecular diffusion. Has been done up to third order closure.

non-local closure the unknown quantity is parameterized by values and/or gradients of known quantities at the *many* points in space. -Analogous to advection processes. Has been done up to first order closure.

2. Local Closure

2a. *Half-order Closure*

Use some of the first moment equations, a variation is called the *bulk method* - that assume a wind or temperature profile and shift it according to the bulk-average background wind or temperature that can be forecast using the prognostic equations for mean quantities (applied to the entire slab).

2b. *First Order Closure*

Retains the prognostic equations for the mean variables, and parameterizes the second moment terms. For example, a dry, horizontally homogeneous BL with no subsidence:

$$
\begin{aligned}
\frac{\partial \overline{U}}{\partial t} &= f_c(\overline{V} - \overline{V_g}) - \frac{\partial \overline{u'w'}}{\partial z} \\
\frac{\partial \overline{V}}{\partial t} &= -f_c(\overline{U} - \overline{U_g}) - \frac{\partial \overline{v'w'}}{\partial z} \\
\frac{\partial \overline{\theta}}{\partial t} &= -\frac{\partial \overline{\theta'w'}}{\partial z}
\end{aligned} \tag{1}
$$

The unknowns are: $\overline{u'w'}$, $\overline{v'w'}$, $\overline{\theta'w'}$.

i. Gradient Transport Theory or K-theory One option is to parameterize the turbulent fluxes of any variable ζ as:

$$
\overline{\zeta' u_j'} = -K \frac{\partial \overline{\zeta}}{\partial x_j} \tag{2}
$$

Where K has units of $m^2 s^{-1}$, for positive K this parameterization implies that $\overline{\zeta' u_j'}$ flows down the local gradient of ζ. This approximation is also calles *small-eddy closure* technique because it fails for larger-size eddies.

Because K relates the turbulent flux to the gradient of the associated mean variable, it is called many names: eddy viscosity, eddy diffusivity, eddy-transfer coefficient, turbulent-transfer coefficient, gradient-transfer coefficient. For statically neutral conditions $K_H = K_E = 1.35 K_m$ where K_E,K_H and K_m are the eddy viscosities for moisture, heat and momentum, respectively.

The equations above would be parameterized as:

$$
\begin{aligned}
\frac{\partial \overline{U}}{\partial t} &= f_c(\overline{V} - \overline{V_g}) + K_m \frac{\partial^2 \overline{U}}{\partial z^2} \\
\frac{\partial \overline{V}}{\partial t} &= -f_c(\overline{U} - \overline{U_g}) + K_m \frac{\partial^2 \overline{V}}{\partial z^2} \\
\frac{\partial \overline{\theta}}{\partial t} &= K_H \frac{\partial^2 \overline{\theta}}{\partial z^2}
\end{aligned} \tag{3}
$$

In an analogous fashion to viscosity, Reynolds stress can by expressed in terms of shear:

$$\tau_{Reynolds} = \rho\overline{u'w'} = \rho K_m \frac{\partial\overline{U}}{\partial z} \tag{4}$$

Turbulence is much more effective than viscosity at causing mixing. $K_m > \nu$
$K_m \approx$ 1 to 10 $m^2 s^{-1}$ - magnitude of the flow
$\nu \approx 1 \times 10^{-5}\ m^2 s^{-1}$ - magnitude of the fluid

Although one can assume that K is constant, this is not a very good approximation.

Figure 1: Figure 6.1 in Book

ii. Mixing-length theory

- Turbulence in a statically neutral environment
- Linear humidity and zonal wind gradient

If turbulence displaces the parcel by a distance z', it will differ from its environment by:

$$\begin{aligned} q' &= -\left(\frac{\partial\overline{q}}{\partial z}\right) z' \\ u' &= -\left(\frac{\partial\overline{U}}{\partial z}\right) z' \end{aligned} \tag{5}$$

For it to have moved in the vertical, it needed a velocity w'.Assume:
$w' = -cu'$ for $\partial\overline{U}/\partial z < 0$
$w' = cu'$ for $\partial\overline{U}/\partial z > 0$

$$w' = c\left|\frac{\partial\overline{U}}{\partial z}\right| z' \tag{6}$$

So now, let's see what the kinematic moisture flux $\overline{w'q'}$ would look like:

$$\overline{w'q'} = -c\overline{z'^2}\left|\frac{\partial \overline{U}}{\partial z}\right|\left(\frac{\partial \overline{q}}{\partial z}\right) \tag{7}$$

We can define a ***mixing length***, l by $l^2 = c\overline{z'^2}$ (makes sense physically), so:

$$\overline{w'q'} = -l^2\left|\frac{\partial \overline{U}}{\partial z}\right|\left(\frac{\partial \overline{q}}{\partial z}\right) = -K_E\left(\frac{\partial \overline{q}}{\partial z}\right) \tag{8}$$

where

$$K_E = l^2\left|\frac{\partial \overline{U}}{\partial z}\right| \tag{9}$$

This tells us that the magnitude of K_E increases as shear increases and as the mixing length increases.

In the surface layer eddies are limited by the earth's surface. It is assumed that $l^2 = k^2z^2$ where k is the von Karman constant, so where

$$K_E = k^2z^2\left|\frac{\partial \overline{U}}{\partial z}\right| \tag{10}$$

Limitations of mixing-length theory:
1. The relationship $w' = cu'$ ***is only valid for statically neutral conditions***.
2. In the atmosphere, gradients are only linear for very small distances - only valid for small eddies.

iii. Other Parameterizations of K Some constraints on the parameterizations of K:

- K=0 when there is no turbulence
- K=0 at the ground
- K increases as TKE increases
- K varies with static stability

- K is non negative (this implies down-gradient transport). However, when there are large eddies associated with rise of warm parcels, the parcels will move regardless of the local gradient. *For this reason, K theory is not recommended for use in convective mixed layers.*

2c. *One-and-a-half Order Closure*

Retains the prognostic equation for the zero order statistics of mean wind, temperature, humidity, and the variances for these variables (TKE for wind variances). Based on the work of Yamada and Mellor (1975), for a dry environment, horizontally homogeneous with no subsidence:

$$\begin{aligned}
\frac{\partial \overline{U}}{\partial t} &= f_c(\overline{V} - \overline{V_g}) - \frac{\partial \overline{u'w'}}{\partial z} \\
\frac{\partial \overline{V}}{\partial t} &= -f_c(\overline{U} - \overline{U_g}) - \frac{\partial \overline{v'w'}}{\partial z} \\
\frac{\partial \overline{\theta}}{\partial t} &= -\frac{\partial \overline{\theta'w'}}{\partial z} \\
\frac{\partial \overline{e}}{\partial t} &= -\overline{u'w'}\frac{\partial \overline{U}}{\partial z} - \overline{v'w'}\frac{\partial \overline{V}}{\partial z} + \frac{g}{\overline{\theta}}\overline{w'\theta'} - \frac{\partial [\overline{w'((p'/\rho) + e)}]}{\partial z} - \epsilon \\
\frac{\overline{\theta'^2}}{\partial t} &= -2\overline{w'\theta'}\frac{\partial \overline{\theta}}{\partial z} - \frac{\partial \overline{w'\theta'^2}}{\partial z} - 2\epsilon_\theta - \epsilon_R
\end{aligned} \tag{11}$$

We add the last two equations because knowledge of TKE and temperature variance gives us an indication of the effectiveness of turbulence and we can use this to improve the parameterizations of eddy diffusivity K_m.

The parameterizations for the unknowns:

$$
\begin{aligned}
\overline{u'w'} &= -K_m(\overline{e}, \overline{\theta'^2})\frac{\partial \overline{U}}{\partial z} \\
\overline{v'w'} &= -K_m(\overline{e}, \overline{\theta'^2})\frac{\partial \overline{V}}{\partial z} \\
\overline{\theta' w'} &= -K_H(\overline{e}, \overline{\theta'^2})\frac{\partial \overline{\theta}}{\partial z} - \gamma_c(\overline{e}, \overline{\theta'^2}) \\
\overline{w'[(p'/\overline{rho}) + e]} &= (5/3)\Lambda_4 e^{-1/2}\frac{\partial \overline{e}}{\partial z} \\
\overline{w'\theta'^2} &= \Lambda_3 e^{-1/2}\frac{\partial \overline{\theta'^2}}{\partial z} \\
\epsilon_R &= 0 \\
\epsilon &= \frac{\overline{e}^{3/2}}{\Lambda_1} \\
\epsilon_\theta &= \frac{\overline{e}^{1/2}\overline{\theta'^2}}{\Lambda_1} \\
K &\approx \Lambda \overline{e}^{1/2}
\end{aligned}
\tag{12}
$$

All Λs represent empirical length-scale parameters often set by trial and error.

Notice that the second correlation terms ar functions of gradients of the mean values, the triple correlations are functions of gradients of second correlations - very similar to first order closure.
Unlike the first-order closure, this closure gives us information about turbulence intensity and temperature variance. The benefits are gained at the expense of increased computational time.

Higher order schemes came with the advent of more computational resources - they are beyond the scope of this class.

3. Non-Local Closure

We will not cover Non-local closure schemes.
The motivation for these schemes is that larger-size eddies can transport fluid across finite distances before the smaller eddies have a chance to cause mixing. It

is an advective perspective that is supported by observations and can account for the upgradient diffusion in the convective boundary layer.