

**Moist Adiabatic Lapse Rate (Γ_M),
 Enthalpy (h),
 Dry static Energy (DSE),
 Moist Static Energy (MSE),
 Equivalent Potential Temperature (θ_e)**

We have described the dry adiabat where an air parcel is lifted rapidly causing the air parcel to expand as the environmental pressure decreases and the air parcel does work on the environment but no heat is exchanged with the environment. As a result some of the air parcel's internal energy is converted to doing work on the environment. The result is the dry adiabatic temperature gradient

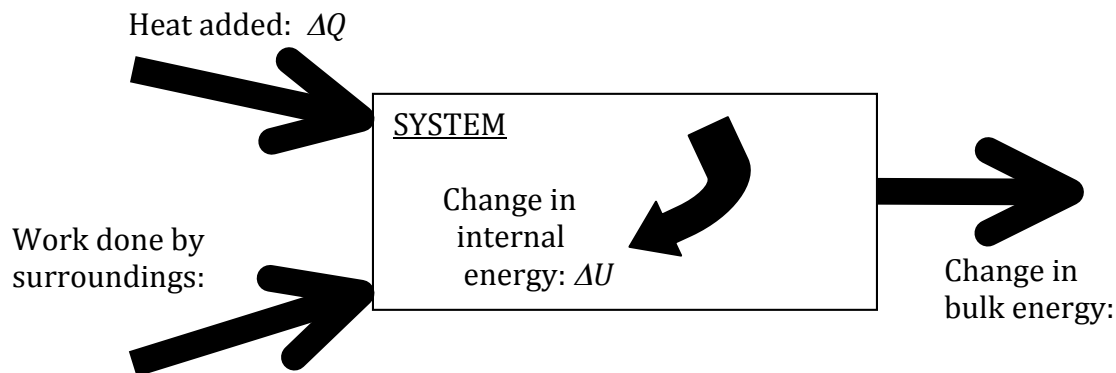
$$\frac{dT}{dz} = -\frac{g}{C_p} \quad (1)$$

Now we discuss the effect of transferring the latent of water vapor via condensation to increase the temperature of the air. This includes discussion about various measures of energy or heat, and introduces the dry and moist static energies (DSE and MSE) and highlights what the concept of enthalpy is.

Also note that in these notes, the variable, q , is used both for the heat per unit mass and the specific humidity.

Recall that when heat enters or is added to a parcel, this heat can go to the internal energy of the parcel, as well as to work the parcel does on the environment. Remember that energy added to the system goes into some combination of the bulk energy of the system, E_B , (which is large scale mechanical energy: kinetic plus potential energy) plus the internal energy of the system, U . We write this as

$$\Delta(U + E_B) = \Delta Q + \Delta W \quad (2)$$



where ΔW is defined as positive when the environment does work on the parcel (**not** when the parcel does work on the environment). Remember also that $\Delta W = -p \Delta V$.

We start with the concept that when heat enters a parcel, this heat can go to the internal energy of the parcel, as well as to work the parcel does on the environment.

Now we ignore changes in bulk energy and express the properties of the parcel in “specific” units (i.e. per unit mass), yielding

$$dq = du - dw \quad (3)$$

Now we take a moment to consider the work term per unit mass.

$$\rho = M_a/V \quad (4)$$

where M_a is the mass of the air parcel and V is the volume of the parcel. The specific volume defined as the volume per unit mass is therefore

$$v = V/m_a = 1/\rho \quad (5)$$

such that the work term per unit volume is

$$-p dV/m_a = -p dv = -d(1/\rho) \quad (6)$$

Therefore we can write (3) which is the energy equation per unit mass as

$$dq = du + p d(1/\rho) \quad (7)$$

Remember that when heat is added to the system (which is the air parcel) when volume (i.e. density) is kept constant and there is no change in bulk energy of the system, the only place for the energy to go is into the kinetic energy of the molecules:

$$dq|_v = du = c_v dT \quad (8)$$

When a gas is heated up from absolute zero temperature, we can define the total specific internal energy per unit mass as

$$u = \int_0^T c_v d\tau = c_v T \quad (9)$$

where the last step assumes that c_v is independent of T . Strictly speaking c_v is a function of temperature as we have discussed (remember that the Boltzmann distribution of the population of the non ground state vibrational modes of the molecules depends on temperature). However, over the range of temperatures encountered in Earth's atmosphere for the gases that make up Earth's atmosphere, this is a good approximation.

Note also that we have assumed that **no phase changes** occurred over this temperature range. We'll address the effects of phase changes in a bit.

Now, when heat is added to the air parcel while it is held at constant pressure (which is the typical situation in the atmosphere), we can rewrite (7) using the ideal gas law:

$$p = \rho \frac{R^*}{m_a} T$$

$$\frac{1}{\rho} = \frac{R^* T}{m_a p} \quad (10)$$

Taking the derivative of both sides of (10) yields

$$d\left(\frac{1}{\rho}\right) = d\left(\frac{R^* T}{m_a p}\right) \quad (11)$$

Substituting (11) into (3) yields

$$dq|_p = du - dw$$

$$\begin{aligned}
&= c_v dT + (R^*/m_a) dT \\
&= [c_v + R^*/m_a] dT \\
&= c_p dT
\end{aligned} \tag{12}$$

As noted previously, there is a work term, because as the gas is heated at constant pressure, it expands and does work on the environment.

Enthalpy

To keep track of the combined internal and work energy per unit mass, we introduce the enthalpy, h , which is defined as

$$h = u + p/\rho = c_v T + R^*/m_a T \tag{13}$$

$$= c_p T \tag{14}$$

Check units of p/ρ : $\text{Pa/m}^2 / (\text{kg/m}^3) = \text{kg m/s}^2 / \text{m}^2 / (\text{kg/m}^3) = \text{kg m}^2 / \text{s}^2 / \text{m}^3 / (\text{kg/m}^3) = \text{J/kg}$

A quicker way to see this is to remember that the units of pressure are the same as the units of energy density. Then the units of p/ρ are $(\text{J/m}^3) / (\text{kg/m}^3) = \text{J/kg}$.

So enthalpy is just the energy associated with the heat capacity (at constant pressure) of the parcel. To be clear, note that enthalpy is not a measure of the *internal* energy contained in a parcel which is $c_v T$. Instead, enthalpy is a hybrid that contains both the internal energy and some of the energy the environment stores to make room for the parcel. In the hydrostatic atmosphere, this second form of energy is stored as the extra gravitational potential of the overriding air that has been lifted to make room for the expanded parcel.

Now consider the equation for dq again and introduce our expression for enthalpy

$$dq = du - dw \tag{3}$$

$$= du + p d(1/\rho)$$

$$= du + \left[d\left(\frac{p}{\rho}\right) - \frac{1}{\rho} dp \right]$$

$$= du + \left[\frac{R^*}{m_a} dT - \frac{1}{\rho} dp \right]$$

$$= c_p dT - \frac{1}{\rho} dp$$

$$dq = dh - \frac{1}{\rho} dp \tag{15}$$

Using the hydrostatic equilibrium equation, we also can write

$$dq = dh + g dz = dh + d\Phi \tag{16}$$

Dry Static Energy

Now the differential equation represented in (16) suggests a new concept of “dry static energy” (DSE) which is defined as

$$DSE = h + \Phi \quad (16)$$

Notice that we can combine (16), (14), the definition of geopotential and the dry adiabat (1) to get

$$\frac{(h + \Phi)}{c_p} = T + \frac{gz}{c_p} = T + \Gamma_d z \quad (17)$$

where Γ_d is the dry adiabatic temperature lapse rate: $\Gamma_d = -dT/dz|_{\text{adiabatic}} = -(g/c_p) = g/c_p$.

Now remember the potential temperature,

$$\theta = T_0 = T_1 \left(\frac{P_0}{P_1} \right)^{\frac{R}{c_p}} = T_1 \left(\frac{P_0}{P_1} \right)^{\frac{R^*}{c_p}} \quad (18)$$

This is the temperature that an air parcel of temperature, T_1 , at pressure, P_1 , would have if it were brought down to the surface ($P_{\text{surf}} = P_0$) dry adiabatically. (18) is written in terms of pressure. We could also write the relation of temperature and potential temperature in terms of altitude in which case

$$\theta = T + \Gamma_d z \quad (19)$$

Therefore, (17) and (19) yield

$$\frac{(h + \Phi)}{c_p} = T + \frac{gz}{c_p} = T + \Gamma_d z = \theta \quad (20)$$

So we see that the DSE divided by the heat capacity at constant pressure is potential temperature. Thus we can keep track of DSE via the potential temperature. Since potential temperature is constant during adiabatic processes, the DSE must also be constant during adiabatic processes.

Expanding the DSE from (16) and (13)

$$DSE = h + \Phi = c_v T + R^*/m_a T + \Phi \quad (21)$$

we also see that the dry static energy is composed of 3 parts:

- (1) the gravitational potential energy of the parcel,
- (2) the internal energy of the parcel, and
- (3) the work term which is the gravitational potential of the air overlying the parcel that has been displaced by the parcel's presence (volume).

We keep track of dry static energy because it is constant during adiabatic processes, when $dq = 0$. Dry static energy goes to zero as a parcel is adiabatically dropped to the ground and then cooled at constant pressure to zero temperature.

DSE and θ are important conceptually because they are conserved in adiabatic processes. Therefore, the DSE and θ of an air parcel do not change when the parcel moves around unless there is heat added to or removed from the parcel.

In some sense, DSE is the natural energy variable to use when the vertical coordinate is height and the potential temperature is the variable to use when the vertical coordinate is pressure.

Moist static energy

Now let's consider what happens when we add the energy associated with phase changes.

As we know, water vapor releases heat when it condenses. The energy released per kg is called the latent heat of vaporization in W+H. You will also see it called the latent heat of condensation. The amount of heat released is a weak function of temperature, with a value of about 2.5×10^6 J/kg at 0°C , and 2.25×10^6 J/kg at 100°C . This is what causes the Clausius Clapeyron equation to be more complicated than if L_v were constant.

The amount of heat released depends on the mass of water vapor condensed

$$dQ = -L_v dm_v \quad (22)$$

or in units of specific heat, we have

$$dQ/m_a = dq = -L_v dm_v/m_a = -L_v dq_v \quad (23)$$

where $q_v = m_v/m_a$ is the specific humidity. It is important to note that there is a minus sign on this latent heat term because vapor content must decrease when condensation occurs.

We can now set this latent heat equal to the heat added to the parcel using (16).

$$dq = dh + d\Phi = -L_v dq_v \quad (24)$$

If there is no heat exchange besides this latent process, we can write

$$0 = dh + d\Phi + L_v dq_v \quad (25)$$

This is called a “**moist adiabatic**” process – the only heat exchange is the conversion of latent heat of vapor into sensible heat as the vapor condenses. Since no heat or mass actually crosses the boundary of our parcel, this is really a conversion of forms of internal energy. We can even consider the un-condensed vapor as a form of internal energy, not as a form of heat.

Analogous to the dry static energy, we can define the “moist static energy” of a parcel as the dry static energy plus the latent heat term

$$MSE = DSE + L_v q_v = h + \Phi + L_v q_v \quad (26)$$

Analogous to the potential temperature, we can define an “**equivalent potential temperature**” that includes the effect of latent heat on potential temperature that provides a way of keeping track of moist static energy,

$$\frac{MSE}{c_p} = \frac{h + \Phi + L_v q_v}{c_p} = \theta + \frac{L_v q_v}{c_p} = \theta_e \quad (27)$$

Note that this represents the temperature that results from conversion of energy to a sensible heat via heat capacity type term.

θ_e is always larger than θ , because it includes the internal potential energy of the vapor. It is also conserved when cloud water condenses or evaporates. You can think of equivalent potential temperature as the temperature a parcel would have if you raised the parcel up to condense all the vapor out, and then brought it down to the surface dry adiabatically which includes the heating associated with compression.

Exercise: What is the potential temperature outside right now? ($T = 34.7$; $T_d = 1.6$; $p = 926$ mb)

In the general adiabatic case for a rising air parcel, $dz > 0$, $dT < 0$ and $dq < 0$. We need an additional constraint on the problem and this is provided by the assertion that $dq = dq_s(T)$,

which is the same thing as saying that RH will max out at 100%. (We will learn later that this is merely an approximation, but it is a good one for coarse purposes). For RH to equal 100% as temperature is decreasing, there must be a corresponding decrease in saturation vapor pressure, which is achieved by condensation.

The saturation specific humidity is the ratio of the saturation density to air density.

$$q_s = \rho_s / \rho_a \quad (28)$$

The density when the water vapor is at saturation is obtained from the saturation vapor partial pressure using the ideal gas law.

$$n_{sv} = e_s(T) / (R^* T) \quad (29)$$

$$\rho_s = n_{sv} m_w = m_w e_s(T) / (R^* T) \quad (30)$$

Using the ideal gas law to relate air density to air pressure, we get

$$\rho_a = p m_a / (R^* T) \quad (31)$$

These lead to the following simplification for q_s

$$q_s = (m_w / m_a) (e_s(T) / p) \quad (32)$$

Now our goal is to find dq_s , which is more easily manipulated via its logarithm...

$$dq_s = q_s \ln q_s = q_s (\ln e_s - \ln p) \quad (33)$$

[Note that we have conveniently neglected the sensitivity of molecular weight of air to vapor pressure, which will contribute a very minor term.]

The first term shows that saturation mixing ratio will increase if saturation vapor pressure increases. Note the second term that shows that saturation specific humidity will increase with a *decrease* in ambient pressure. This might seem counterintuitive, since we know that q doesn't change with changes in pressure. However, q_s is a different beast than q . Whereas q keeps track of the mass of vapor in a given mass of air, q_s tracks the mass of water vapor that would be in equilibrium with a flat surface of water per unit mass of air. As pressure increases and decreases, saturation vapor pressure doesn't change because it is only sensitive to temperature. However, the dry air mass is fluctuating with pressure, causing the constant saturation vapor pressure to be a relatively larger fraction of the air pressure as pressure goes down.

To better understand this, consider a chamber that is partially filled with water and is held at temperature T . In equilibrium, the total pressure will be $p_d + e_s(T)$, where p_d is the pressure due to the dry air. Now we remove half the air in the chamber. At this instant, the pressure will be $p_d/2 + e_s(T)/2$, and q will be the same as before since there has been no condensation or evaporation (yet). But the water surface will still have a saturation vapor pressure of $e_s(T)$. So over time, water will evaporate and q will increase to a new, higher equilibrium value, $q_s(T, p_f)$.

Now back to the moist parcel that is ascending adiabatically. From Clausius-Clapyron, we have

$$\frac{de_s}{e_s} = d \ln e_s = \frac{L m_v}{R^* T} \frac{dT}{T} = \frac{L}{R_v T} \frac{dT}{T} \quad (34)$$

The change in hydrostatic pressure for ideal gases can be written as

$$d \ln p = -dz/H \quad (35)$$

where the scale height of the atmosphere is $H = R^*T/(m_a g)$. Thus our expression for dq_s can be written in terms of dT and dz , which are the same independent variables used in our dry adiabatic equation.

$$dq_s = q_s (d \ln e_s - d \ln p) = q_s \left(\frac{L_c m_v}{R^* T} \frac{dT}{T} + \frac{dz}{H} \right) \quad (36)$$

Subbing all this into the energy conservation equation yields,

$$0 = dh + d\Phi + L_v dq_v \quad (25)$$

$$0 = c_p dT + g dz + L_c q_s \left(\frac{L_c m_v}{R^* T} \frac{dT}{T} + \frac{dz}{H} \right) \quad (37)$$

Now we can solve for $dT/dz = -\Gamma_M$, which is called the moist adiabatic lapse rate...

$$\frac{dT}{dz} = - \frac{g + L_c q_s / H}{c_p + L_c q_s \left(\frac{L_c m_w}{R^* T^2} \right)} = -\Gamma_M \quad (38)$$

At this point, it is useful to note the g/c_p is the dry adiabatic lapse rate, Γ_D , and that H is a function of gravity. After some algebra, we get,

$$\Gamma_M = \Gamma_D \frac{1 + \frac{L_c q_s M_A}{R^* T}}{1 + \frac{L_c q_s}{c_p T} \frac{L_c M_w}{R^* T}} \quad (39)$$

When all is said and done, we see that there are two terms that make the moist adiabatic lapse rate different from the dry value. The numerator contains a term that came from the pressure dependence of saturation specific humidity (as discussed above). The denominator has a term that came from the temperature dependence of saturation specific humidity through the Clausius Clapeyron relation, and this is the dominant term. While both terms are important to getting the right value, the CC term dominates.

At 0°C and mean sea level pressure (MSL) the denominator is 1.68 and the numerator is 1.12, with that the moist lapse rate is (6.5 K/km), about 2/3 the value of the dry lapse rate. At 15°C, the numerator is 1.312 and the denominator increases to 2.630, for a moist lapse rate of 4.9 – half the value of the dry lapse rate.

This addition of the 2nd term in the denominator signifies that that both kinetic and latent energy sources are used to do work on the environment as it expands, thus slowing the cooling relative to the case when only kinetic energy is used. The modifier in the numerator accounts for the evaporation (and hence cooling) that happens independently of temperature as air depressurizes, due to the drop in vapor pressure at fixed specific humidity. Because it represents a cooling with height, it increases the magnitude of the lapse rate, but only slightly relative to the CC effect.

