Assuming the linear function $\hat{y} = a_1 x + a_0$

$$\%_{\text{exp var}} = \frac{\text{Exp. var.}}{\text{Tot. var.}} = \frac{\Sigma (\hat{y} - \bar{y})^2}{\Sigma (y - \bar{y})^2} = \frac{\Sigma (a_1 x + a_0 - \bar{y})^2}{\Sigma y'^2}$$

$$= \frac{\Sigma (a_1 x + \bar{y} - a_1 \bar{x} - \bar{y})^2}{\Sigma y'^2} = \frac{\Sigma (a_1 x')^2}{\Sigma y'^2}$$

Substitute for a_1

$$= \frac{\Sigma \left(\frac{x'y'}{x'^2}\right)^2 \Sigma x'^2}{\Sigma y'^2}$$

$$= \frac{\Sigma (x'y')^2}{\Sigma y'^2 \Sigma x'^2}$$

Divide by N to get

$$\%_{\text{exp. variance}} = \boxed{\frac{\overline{(x'y')}^2}{\overline{y'^2}\ \overline{x'^2}} = r^2}$$

In words, the square of the covariance divided by the variances is equal to the explained variance.

the correlation coefficient is just the square root of the explained variance.

$$r = \frac{\bar{x}^T \bar{y}'}{\sqrt{\bar{x}'^2}\sqrt{\bar{y}'^2}} = \frac{\bar{x}^T \bar{y}'}{\sigma_x \sigma_y} = \frac{\sigma_{xy}}{\sigma_x \sigma_y}$$

Properties of r^2 and r.

1) r^2 is the % variance explained by linear fit $\hat{y}$

2) r^2 always is between 0 and 1

3) r varies from -1 to 1

$r > 0$ → positively correlated (variables relate in the same way)

$r < 0$ → negatively correlated (variables relate in the opposite way)

Relationship of r to a_1

$$a_1 = r \cdot \frac{\sigma_y}{\sigma_x}$$

Correlation coef. × ratio of std. deviations.

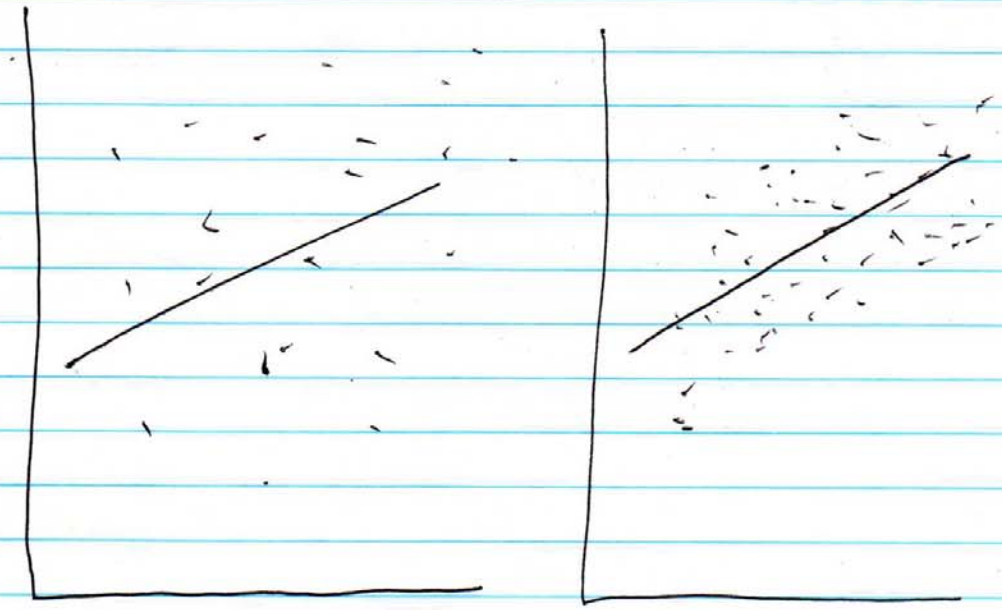
What r and a_1 give physically

$$a_1 = \frac{\overline{x'y'}}{\overline{x'^2}}$$ Change in y per change in x

$$r = \frac{\overline{x'y'}}{\sigma_x \sigma_y}$$ How good the regression fit is (the spread)

So a large value in a_1 does not necessarily mean a large correlation coefficient.

Consider the following scatter plots with the same regression line (a_1)



Weakly correlated
Lots of scatter

Highly correlated
Tightly clustered.

The quantity $\sqrt{1-r^2}$ has a special physical meaning too:

$1-r^2$ → Unexplained variance

$\sqrt{1-r^2}$ → Root mean square error (RMSE)

CAVEATS ON CORRELATION!

- Only works for linear relationships
- Does not reveal quadrature relationships (when things are 90° out of phase)
- We've assumed the data (xi, yi) are linearly independent (i.e. not autocorrelated) → A big one!!
- Correlation DOES NOT ESTABLISH PHYSICAL CAUSE AND EFFECT
  e.g. I think this is a major handicap of any statistically-based forecasting technique.
  Typically find some "cool" relationship between predictor(s) and predictand(s), establish a methodology that works in hindcast mode, apply in operational forecasting.

  Hurricane forecasting: Bill Gray bases forecast on things like
  - QBO
  - African rainfall
  - Atlantic SST
  - El Niño

  } How do these physically interact to give more or less hurricanes? Stats. don't say...

Quote from Bill Gray at my master's defense: "Why do want to waste your time on modeling?"

My perspective: A sound and robust approach to physical understanding + predictability should include stat. & physically based methods - the latter is next semester!!

Sampling theory of correlation.

- When we calculate the sample correlation coefficient r, we are estimating the correlation coefficient assuming an infinite sample size (ρ). ρ is the "true" correlation coefficient.

Case $\rho = 0$

- When the true correlation coefficient is zero, then the sampling distribution of r will follow the t-distribution.

~~Statistic~~ is:

$$t = \frac{r\sqrt{N-2}}{\sqrt{1-r^2}}$$

Student's t-distribution with $\nu = N-2$.

Example: $N = 10$ $r = 0.6$.

Does r differ from $\rho = 0$ at the 95% level?

$$t = \frac{r\sqrt{N-2}}{\sqrt{1-r^2}} = 2.12.$$

2-tailed test (95%)

$t_{crit} = t_{0.975} = 2.31$

X Not satisfied.

1-tailed test is satisfied

(H_0)

∴ State that the null hypothesis that $\rho = 0$ cannot be rejected against alternative hypothesis that $\rho \neq 0$ (H_1).

When to use 1-sided vs. 2-sided test:

1-sided: Have a prior expectation that correlation should be positive or negative.

2-sided: No prior expectation that correlation should be of either sign.

Case $\rho \neq 0$

When the true correlation coefficient is not expected to be zero, distribution for $\rho \neq 0$ is skewed.

Use Fisher's z-transformation to convert distribution of r to normal.

$$z = \frac{1}{2} \ln\left\{\frac{1+r}{1-r}\right\} \rightarrow \text{z-statistic}$$

This statistic is normal with mean:

$$\mu_z = \frac{1}{2} \ln\left\{\frac{1+\rho_0}{1-\rho_0}\right\}$$

Std. deviation

$$\sigma_z = \frac{1}{\sqrt{N-3}}$$

Example from Hartmann:

$N = 21 \quad r = 0.8 \Rightarrow$ What is 95% conf. interval?

$$z = \frac{1}{2} \ln \left\{ \frac{1+0.8}{1-0.8} \right\} = 1.098$$

$$z - 1.96\sigma_z < \mu_z < z + 1.96\sigma_z$$

$$0.6366 < \mu_z < 1.560.$$

Transforming this back in terms of correlation:

$$0.56 < \rho < 0.92 \rightarrow 95\% \text{ conf. interval.}$$

Also a similar test for differences in correlation, similiar to t-test.

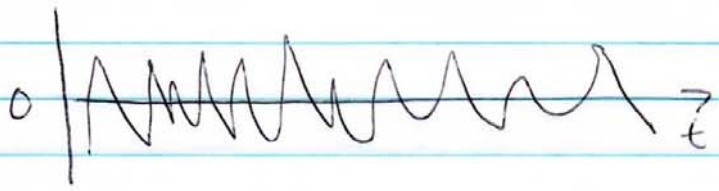
How to account for autocorrelation in evaluating statistical significance?

→ Problem: Data may not be independent in time. *This persistance reduces the degrees of freedom in the data.

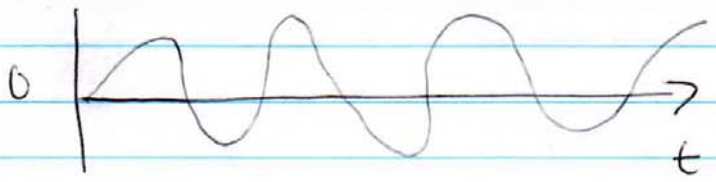
Example mentioned before:
ENSO time series (of whatever index) monthly for 50 years. Is the true sample size = 50 years × 12 $\frac{\text{months}}{\text{yr}}$ = 600 months?

→ Answer: NO because sea surface temperatures in the tropical Pacific are fairly persistent from month to month.

The more autocorrelated data is, the "smoother" it will appear in time:



Little correlation in time
"White"



More correlation in time
"Red"

Various ways to account for the autocorrelation
Simplest way is just to use lag-1 autocorrelation.

Lag-1 Autocorrelation

→ time

Original Data: a_1 a_2 a_3 a_4 ... a_n
Lagged Data: a_1 a_2 a_3 ... a_{n-1} a_n

Calculating the correlation of the following data series: (a_i, a_{i-1})
where i = time index

This correlation is called lag-1 autocorrelation and is denoted by ρ_1.

Use ρ_1 to get a modified sample size:

Effective sample size →

$$n' \cong n \frac{1-\rho_1}{1+\rho_1}$$

n = original # of samples

n' = # of samples accounting for persistence.

where $n' < n$

The effective sample size can also be used to inflate the variance accounting for autocorrelation

Adjusted variance →

$$\frac{s^2}{n'} = \frac{s^2}{n}\left(\frac{1+\rho_1}{1-\rho_0}\right)$$