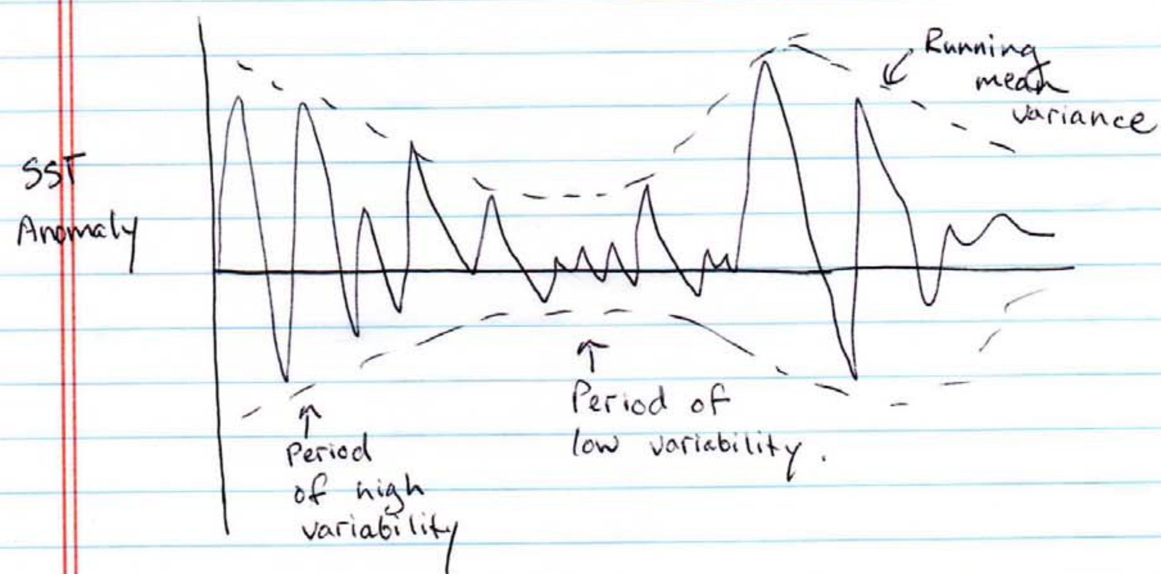
Wavelet Analysis

Ref: Torrence and Compo BAMS Article
"A Practical Guide to Wavelet Analysis"
+ web references (has code too!)

Motivation: Many time series exhibit non-stationarity in their statistics.

A series may contain periodic signals, but these may vary in frequency and time (and typically do!)

Niño example



If we do just a plain spectral analysis, will completely miss how the spectral power is changing with time!

Could just look a specific time windows for Fourier analysis...

Disadvantages:

1) The choice of the length of the window is arbitrary and will have windowing issues. Ideally want to allow for variety of scales.

2) Low frequencies are lost by chopping up the time series.

3) Assumes the signal can be broken down into sines and cosines.

Idea behind wavelets:

Decompose time series into time / frequency space simultaneously. So have 2 pieces of info:

- Amplitude of periodic signals
- How amplitude varies with time (phase)

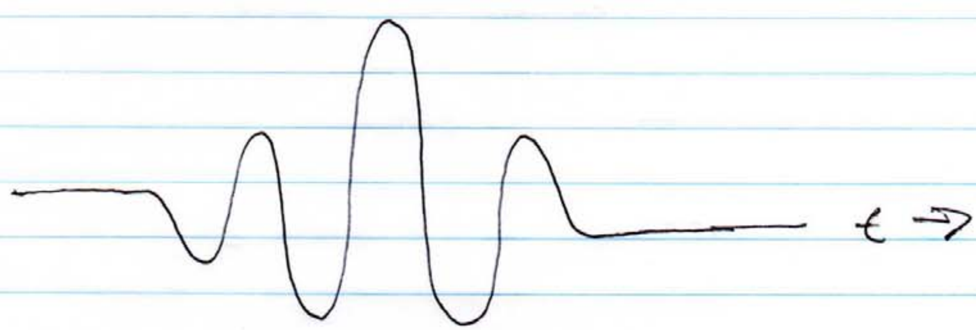
Wavelet function replaces the sinusoidal functions.

Wave packet (for an individual wavelet) has finite duration and a specific frequency.

Considerations for wavelets:
- Orthogonal or not orthogonal
- Complex or real
- Width
- Shape.

Many types of wavelets. Examples: Morlet, Mexican Hat (looks like sombrero), Paul.

Most commonly used is Morlet



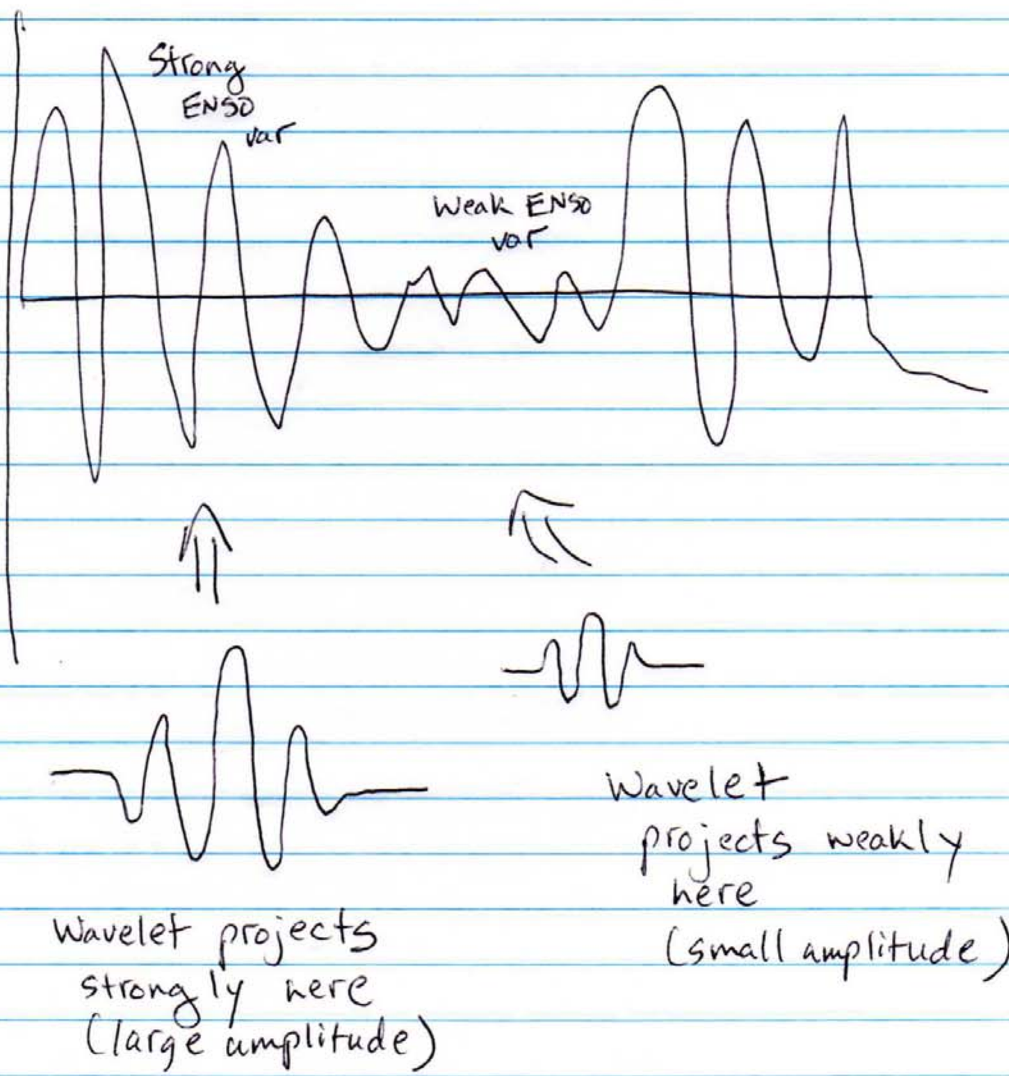
Characteristics of Morlet wavelet:

- Just a sine wave multiplied by a Guassian. (center of wave corresponds to peak in bell curve)

- Width of the wavelet corresponds to the desired frequency. wider, the longer period is.

"Slide" the wavelet along the time series to construct a new time series of the projection amplitude vs. time.

Back to Nino example
Wavelet with corresponding period = 5 yrs.



Equation for the Morlet wavelet

$$\psi_0(\eta) = \pi^{-1/4} \, e^{-\eta^2/2} \, e^{i\omega_0 \eta}$$

Gaussian part   Wave part

η = (non-dimensional) time
ω_0 = wave number (i.e. width of wavelet)
$\psi_0(\eta)$ = Wavelet function.

Since the wavelet function is complex, it contains both amplitude + phase info.

Wavelet transform ($W_n(s)$)

For a (time) series x_n: (n = time)

$$W_n(s) = \sum_{n'=0}^{N-1} x_n \, \psi^* \left[\frac{(n' - n)\,\delta t}{s} \right]$$

The convolution of a time series with a scaled and translated version of the wavelet.

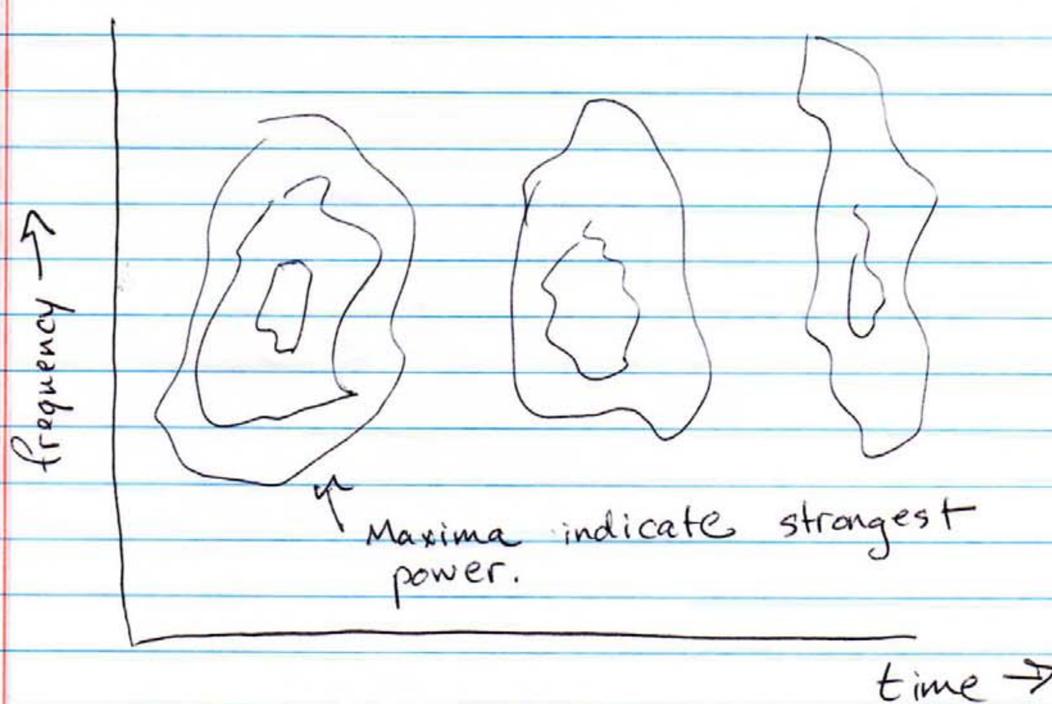
s = given frequency scale.
($>$ lowest possible frequency & often multiple of it)
n = localized time index.

From this can construct a two-dimensional picture of the wavelet transform as a function of frequency and time.

Plot $W_n^2(s)$: Power spectrum

$W_n(s)$: Actual oscillations of the wavelets with phasing.

<u>Wavelet power spectrum</u>



The power spectrum is typically normalized by the total variance of the time series (σ^2)

Other notes

Cone of influence: Region of wavelet spectrum in which edge effects become important. Defined via e-folding time for autocorrelation each freq. scale. Typically shown as hatched region.

Red noise spectrum

Most rigorous way is via Monte-Carlo technique:
- Construct many, many AR(1) time series (i.e. using a Markov model)
- Compute wavelet power spectra for each time series.
- Avg. all the results to get background red noise spectrum.

(Best to do this way because it accounts for χ^2 dist)

Significance

Similar to spectral analysis, multiply the red noise spectrum by χ^2 value for the desired confidence interval (div. by 1/2 to remove dof factor)

Global = values obtained by averaging data from wavelet analysis over given freq. for all times.