Summary of EOF analysis via
eigenanalysis of the covariance matrix

1) If data are in the form $M \times N$ where M = time, N = space, calculate the temporal covariance matrix $(A^T A)$

2) Diagonalize $A^T A$. The 1st EOF is the eigenvector with highest λ, etc.
(need a computer...)

3) Find PCs by projecting A onto the eigenvectors E.

4) The fraction of variance explained by each EOF/PC pair is:

$$\frac{\lambda_i}{\sum_{i=1}^{N} \lambda_i} = \frac{\text{Eigenvalue of given EOF}}{\text{Sum of all eigenvalues}}.$$

No other combination of k predictors can explain a larger fraction of the variance than the 1st k PCs.

Other notes - Interchangability.

* In case of AA^T, the first eigenvector is the leading PC of A.

$\left.\begin{matrix} {}^*A\,A^T \\ A^T A \end{matrix}\right\}$ share the same eigenvalues

EOFs and PCs are completely interchangable with regard to the matrix manipulation.

Using SVD to calculate EOFs / PCs

SVD = Singular Value Decomposition

Why use this method?

→ It is how most EOF/PC routines that are canned computer programs work.

→ Code for it is straightforward, easily accessible. A single command in Matlab!

→ If we understand basically what the SVD routine is doing, it's fine to use it as a "black box" and focus our attention to physical interpretation of the results.

How does it work?

Any rectangular matrix can be represented as the product of 3 special matrices.

$$A\,(M \times N) = U\,(M \times M) \cdot \Sigma\,(M \times N) \cdot V\,(N \times N)$$

"M" = time
"N" = space.

This is denoted in matrix notation

$$\boxed{A = U \Sigma V^T}$$

The decomposition of A into U, Σ, V^T must satify the following relationships:

(1) $AA^T u_i = \sigma_i u_i \Rightarrow C_{M\times M} u_i' = \sigma_i u_i'$

(2) $A^T A v_i = \sigma_i v_i \Rightarrow C_{N\times N} v_i' = \sigma_i v_i'$

Where u_i and v_i are the i^{th} columns of matrices U and V, respectively.

<u>Physical interpretation</u>

(1) → The eigenvectors (u_i) of the covariance matrix based on the time dimension (M)

Columns in U matrix (u_i) = Eigenvectors of AA^T = PCs of A.

(2) → The eigen vectors (v_i) of the covariance matrix based on the space dimension (N).

Columns in V matrix (v_i) = Eigenvectors of $A^T A$ = EOFs of A.

(3) → The σ_i values correspond to the square roots of the eigenvalues (λ_i)
(WILL PROVE THIS LATER...)

In the case $M = N$ and A is full rank.

$$\Sigma = \begin{array}{c} \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ & & \ddots \\ 0 & 0 & \sigma_N \end{bmatrix} M \\ N \end{array}$$

If $M > N$

$$\Sigma = \begin{array}{cc} \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ & & \ddots \\ 0 & 0 & \sigma_N \\ 0 & 0 & 0 \end{bmatrix} & \begin{array}{c} \uparrow \\ N \\ \downarrow \\ N+1 \\ \downarrow \\ M \end{array} \quad M \\ N & \end{array}$$

Null space.

<u>Notes on "i"</u>

The largest possible rank of A is $q = \min(M, N)$
All columns in U (or V) where $i \geq q$ lie in the nullspace of A.

Additional notes:

- Columns in U and V matrices (u_i and v_i) must be orthogonal
- Columns in U and V span R^m and R^N, respectively
- Columns in u, v are orthonormal.

What about Σ matrix?

Σ matrix can be viewed as comprising 2 matrices, depending on the relative size of $M \times N$.

For $M > N$
$$\Sigma = \begin{array}{cc} \leftarrow N \rightarrow & \\ \begin{bmatrix} D \\ 0 \end{bmatrix} & \updownarrow M \end{array}$$

$D = N \times N$ diagonal matrix

$0 = (M-N) \times N$ matrix of zeros.

The first q elements (where $q =$ rank of A) of D are the singular values of A. Singular values are the square roots of the eigenvalues of A. ($\sigma_i^2 = \lambda_i$)

The $q > N$ diagonal elements are zeros, corresponding to the nullspace of A.

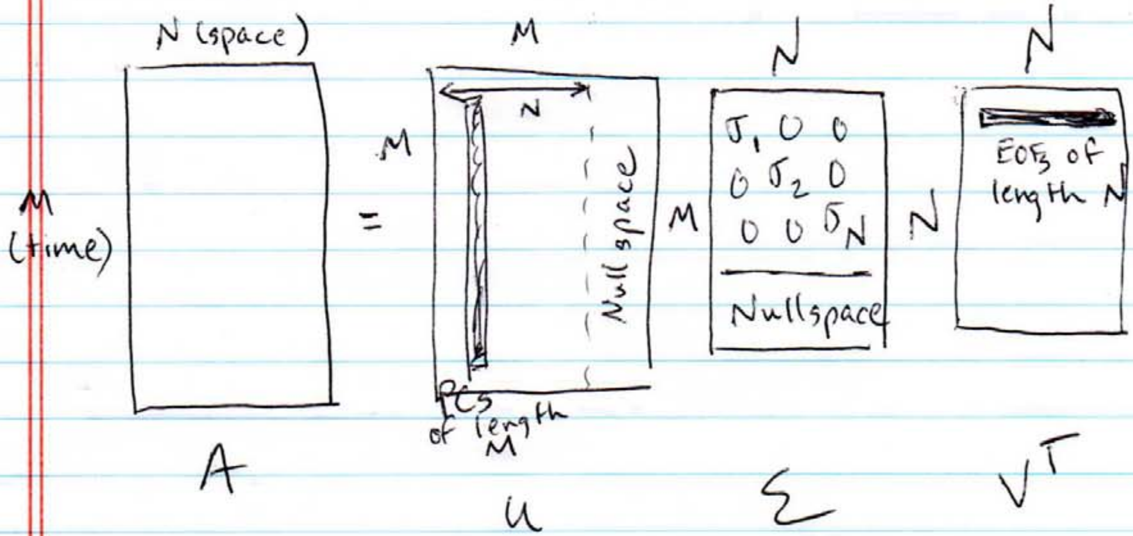
Example

If $M > N$ and A is full rank.

* Reverse the logic if $N < M$, which is typically the case...

- U spans R^m; V spans R^N
- N columns in V span R^N
- 1^{st} N columns in U span the column space of A.
- $N+1 \rightarrow M$ columns in U span the null space of A.

Graphically



$u_1 \rightarrow u_N$ spanning A's column space (PCs)

$u_{N+1} \rightarrow u_M$ Null space not representable in terms of A's columns

$V_1 \rightarrow V_N$ spans A's row space (EOFs)

Physical interpretation

If the σ's are arranged such that σ_1 is the largest value:

- $\rightarrow$ 1^{st} column in U is the leading PC of A.
- $\rightarrow$ 1^{st} column in V is the leading EOF of A.

EOF modes

$$A = U \Sigma V^T$$

Can be written

$$A = \sum_{i=1}^{N} \sigma_i u_i v^T$$

Schematically:

$$\underset{(M \times N)}{A} = \sigma_i \underbrace{\begin{bmatrix} u_1 \\ \downarrow \end{bmatrix} \begin{bmatrix} v_1 \longrightarrow \end{bmatrix}}_{\text{EOF mode 1}} + \cdots$$

$$\sigma_N \underbrace{\begin{bmatrix} u_N \\ \downarrow \end{bmatrix} \begin{bmatrix} v_N \longrightarrow \end{bmatrix}}_{\text{EOF mode } N} \cdots$$

Equivalence of SVD to Eigenanalysis (or diagonalization) of covariance matrix. (proof)

SVD: $A = U \Sigma V^T$

Square both sides of $A = U \Sigma V^T$

$$A^T A = V \Sigma^T U^T U \Sigma V^T$$

$U^T U = I$ $\qquad V^T V = I$ (since they're symmetric)

$$A^T A = V \Sigma_N^2 V^T$$

or equivalently, since $A^T A$ = Cov. matrix

$$C V = V \Sigma_N^2$$

$$C V = V \lambda$$ → Original eigenvalue problem

Cov. matrix $\qquad$ Eigen-vectors $\qquad$ Eigenvalues.

Relationship between EOFs / PCs.

$A = U \Sigma V^T$

Multiply by V

$AV = U \Sigma$

Hence, for an individual mode:

$A v_i = \sigma_i u_i$

Physically: PCs are found by projecting the data on to the EOFs and vice versa.

Summary of finding EOFs / PCs via SVD:

1) Decompose matrix A such that

$A = U \Sigma V^T$

2) for A $(M \times N)$
if M is the sample space:
- EOFs (spatial patterns) correspond to the columns in V
- PC (time series) correspond to the columns in U.

3) The fraction of variance explained by the i^{th} EOF / PC pair is:

$$\frac{\sigma_i'^2}{\sum_{i=1}^{N} \sigma_i'^2} \quad \text{or} \quad \frac{\lambda_i'}{\sum_{i=1}^{N} \lambda_i'}$$

Presentation of EOFs

Whether the EOFs or PCs are shown depends on which domain yields interesting structure.

Problem with looking directly at EOFs is that they are normalized, so don't get a sense of physical units.

Common way to show the EOF spatial patterns is to regress the data (i.e A matrix) onto the standarized PC time series.

Steps

1) Get the PC { via SVD
projecting A onto EOFs via diag. method

2) Standardize the PC so it has a variance = 1 (unit variance)

$$PC = \frac{PC}{\sqrt{\overline{PC'^2}}} \rightarrow \text{std. dev. of PC.}$$