where C_{ij} is the the inner product between row $x(i,:)$ and column $y(:,j)$

$$C_{M \times N} = X_{M \times K} \, Y_{K \times N}$$

K is the inner dimension consumed by the multiplication.

* This is not commutative $(xy \neq yx)$

Wilks' example of matrix multiplication (p. 413)

$$\begin{array}{ccc} X & Y & C \\ \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \end{bmatrix} & \begin{bmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \\ b_{3,1} & b_{3,2} \end{bmatrix} = & \begin{bmatrix} c_{1,1} & c_{1,2} \\ c_{2,1} & c_{2,2} \end{bmatrix} \\ 2 \times 3 & 3 \times 2 & 2 \times 2 \end{array}$$

$$C = \begin{bmatrix} c_{1,1} & c_{1,2} \\ c_{2,1} & c_{2,2} \end{bmatrix} = \begin{bmatrix} \cancel{a_{1,1}b_{1,1} + a_{1,2}b_{2,1} + a_{1,3}b_{3,1}} \end{bmatrix}$$

(X 1st row - 1st column Y) (1st row X - 2nd col. Y)

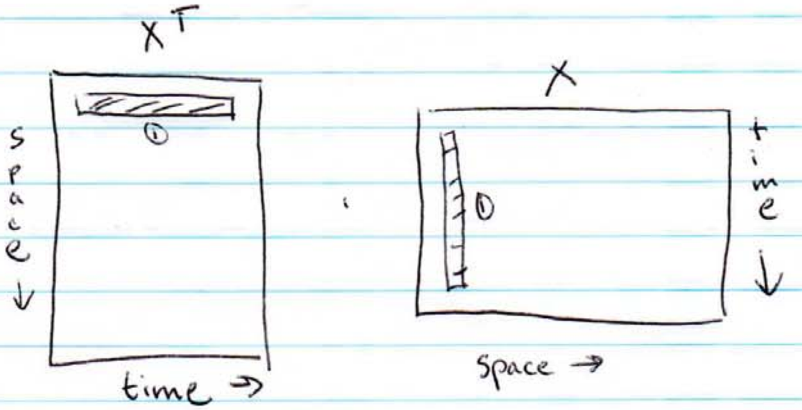
$$= \begin{bmatrix} a_{1,1}b_{1,1} + a_{1,2}b_{2,1} + a_{1,3}b_{3,1} & a_{1,1}b_{1,2} + a_{1,2}b_{2,2} + a_{1,3}b_{1,3} \\ a_{2,1}b_{1,1} + a_{2,2}b_{2,1} + a_{2,3}b_{3,1} & a_{2,1}b_{1,2} + a_{2,2}b_{2,2} + a_{2,3}b_{3,2} \end{bmatrix}$$

(2nd row X - 1st col. Y) (2nd row X - 2nd col Y)

The Covariance Matrix (IMPORTANT!)

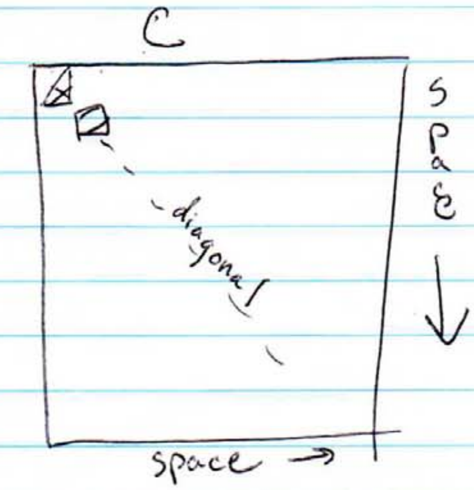
Consider data that has a time mean of zero.

Data matrix X that is $M \times N$ where M is time, N is space.



$(N \times M)$ $(M \times N)$

Result . . .



$(N \times N)$

Yields the the covariance matrix (C)*

$$\boxed{X^T X = C}$$

* multiplied by the number of values (N)

Properties of the covariance matrix:

- Diagonal elements: Sample variances of the N variables in space.
- off diagonal elements: Covariances among all possible pairs of N variables in space.

Correlation matrix is obtained if the time elements (columns) have a mean = 0 and a std. dev. = 1.

Why is this important?

→ Eigenvalue analysis of the covariance matrix will yield the most statistically dominant spatially varying patterns (ie. EOFs)

We'll get to that part a little later ...

Vector spaces and rank $(R_{\#})$

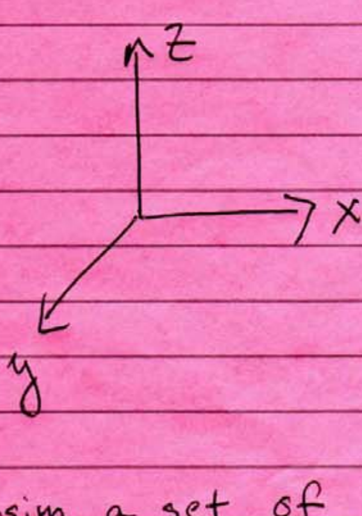
R^0 is a single point
R^1 is a real line
R^2 is where all 2 element vectors reside: $[x_1, x_2]$ is the vector stretching from $(0,0)$ to (x_1, x_2)
R^3 is the 3-D space: $[x_1, x_2, x_3]$ stretches from $(0,0,0)$ to (x_1, x_2, x_3)

Vector spaces must be "spanned" by a series of "basis" vectors.

A basis for R^N is a set of N vectors which can be used to specify any point in R^N.

E.g. There are infinite bases for spanning R^3. The most common basis is the vectors:

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$
$$\downarrow \qquad \downarrow \qquad \downarrow$$
$$x \qquad y \qquad z$$



The only constraint when choosing a set of basis vectors is that the vectors are linearly independent $\rightarrow$ no vector corresponds

to a linear combination of other vectors.
(This defines orthongonality)

Say you have

$$\alpha e_1 + \beta e_2 + \chi e_3 = 0$$

Where e_1, e_2, and e_3 are basis vectors (orthogonal)

These would satisfy:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = 0.$$

What about this example...

$$i = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad j = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad k = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

$\mathbb{R}^3$ is not spanned. k is a linear combination of i & j (i.e. not orthogonal)

Therefore, the orthogonal vector k would be perpindicular to the plane of i & j.

Fundamental spaces and rank

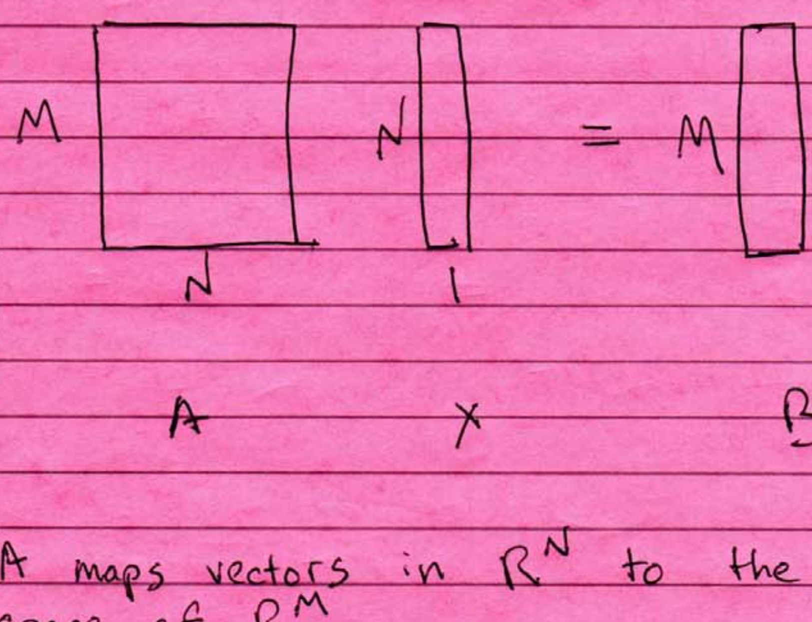
Matrices are associated with 2 fundamental spaces:

- column space
- row space

Consider:

$$Ax = b$$

A = Matrix of $M \times N$
x = Vector of N (basis vectors)
b = Vector of M



A maps vectors in R^N to the parallel space of R^M

B is the projection of the rows of A into the parallel space of x.

B is simply a linear combination of A's column:

$$B = A(:,1) \cdot x_1 + A(:,2)\, x_2 + \dots A(:,N)\, x_N$$

* If you want to determine whether a set of basis vectors span R^N, need to prove there is no x that satisfies $Ax = 0$, where the columns in A are the basis vectors. (i.e. then have orthogonality).

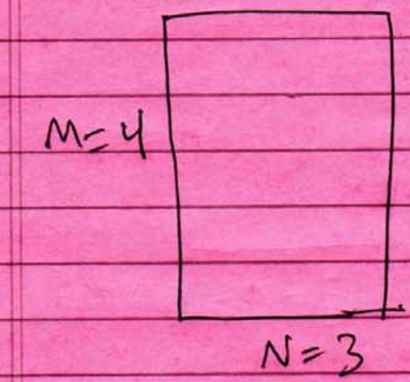
If no x satisfies $Ax = 0$, then the rows and columns in matrix ~~A~~ are linearly independent.

(Will show an example of this a bit later to illustrate the point...)

<u>Rank</u> (q): the vector space spanned by a matrix. The most it can be is the smallest dimension of the matrix.

$$\boxed{q \leq \min(M,N) \quad \text{for } M\times N \text{ matrix}}$$

e.g. Have $M = 4$; $N = 3$



This matrix at most spans $\mathbb{R}^3$.

Key word is "at most". Oftentimes matrices do not have a rank that corresponds to their smallest dimension → it is less!

Why is this?

→ Oftentimes the requirement that no x satisfies $Ax = 0$ is not satisfied.

→ In other words, one basis vector can be expressed as a combination of others and they are not orthogonal.

Consider this example:

$$A = \begin{bmatrix} 1 & 1 & 4 \\ 1 & 2 & 3 \\ 1 & 3 & 2 \\ 1 & 4 & 1 \end{bmatrix}$$

Is this matrix $\mathbb{R}^3$?